**Last Lecture**

- What are the typical steps for constructing a likelihood ratio test?
- Is LRT statistic based on sufficient statistic identical to the LRT based on the full data?
- When multiple parameters need to be estimated, what is the difference in constructing LRT?
- What is unbiased test?

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**Unbiased Test**

**Definition**

If a test always satisfies

\[
\Pr(\text{reject } H_0 \text{ when } H_0 \text{ is false }) \geq \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is true })
\]

Then the test is said to be unbiased.

**Alternative Definition**

Recall that \( \beta(\theta) = \Pr(\text{reject } H_0) \). A test is unbiased if

\[
\beta(\theta') \geq \beta(\theta)
\]

for every \( \theta' \in \Omega_0^c \) and \( \theta \in \Omega_0 \).
Example

\( X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2) \) where \( \sigma^2 \) is known, testing \( H_0 : \theta \leq \theta_0 \) vs \( H_1 : \theta > \theta_0 \).

LRT test rejects \( H_0 \) if \( \frac{\overline{X} - \theta_0}{\sigma / \sqrt{n}} > c \).

\[
\beta(\theta) = \Pr\left( \frac{\overline{X} - \theta_0}{\sigma / \sqrt{n}} > c \right) \\
= \Pr\left( \frac{\overline{X} - \theta + \theta - \theta_0}{\sigma / \sqrt{n}} > c \right) \\
= \Pr\left( \frac{\overline{X} - \theta}{\sigma / \sqrt{n}} + \frac{\theta - \theta_0}{\sigma / \sqrt{n}} > c \right) \\
= \Pr\left( \frac{\overline{X} - \theta}{\sigma / \sqrt{n}} > c - \frac{\theta - \theta_0}{\sigma / \sqrt{n}} \right)
\]

Note that \( X_i \sim N(\theta, \sigma^2), \overline{X} \sim N(\theta, \sigma^2/n), \) and \( \frac{\overline{X} - \theta}{\sigma / \sqrt{n}} \sim N(0, 1) \).

Example (cont’d)

Therefore, for \( Z \sim N(0, 1) \)

\[
\beta(\theta) = \Pr\left( Z > c + \frac{\theta_0 - \theta}{\sigma / \sqrt{n}} \right)
\]

Because the power function is increasing function of \( \theta \),

\( \beta(\theta') \geq \beta(\theta) \)

always holds when \( \theta \leq \theta_0 < \theta' \). Therefore the LRTs are unbiased.

Uniformly Most Powerful Test (UMP)

**Definition**

Let \( \mathcal{C} \) be a class of tests between \( H_0 : \theta \in \Omega \) vs \( H_1 : \theta \in \Omega \). A test in \( \mathcal{C} \) with power function \( \beta(\theta) \) is uniformly most powerful (UMP) test in class \( \mathcal{C} \) if \( \beta(\theta) \geq \beta'(\theta) \) for every \( \theta \in \Omega \) and every \( \beta'(\theta) \), which is a power function of another test in \( \mathcal{C} \).

UMP level \( \alpha \) test

Consider \( \mathcal{C} \) be the set of all the level \( \alpha \) test. The UMP test in this class is called a UMP level \( \alpha \) test.

UMP level \( \alpha \) test has the smallest type II error probability for any \( \theta \in \Omega \) in this class.

- A UMP test is "uniform" in the sense that it is most powerful for every \( \theta \in \Omega \).
- For simple hypothesis such as \( H_0 : \theta = \theta_0 \) and \( H_1 : \theta = \theta_1 \), UMP level \( \alpha \) test always exists.
Neyman-Pearson Lemma

**Theorem 8.3.12 - Neyman-Pearson Lemma**

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding to $\theta_i$ is $f(x|\theta_i)$, $i = 0, 1$, using a test with rejection region $R$ that satisfies

- $x \in R$ if $f(x|\theta_1) > kf(x|\theta_0)$ (8.3.1) and
- $x \in R^c$ if $f(x|\theta_1) < kf(x|\theta_0)$ (8.3.2)

For some $k \geq 0$ and $\alpha = \Pr(X \in R|\theta_0)$, Then,

- ( Sufficiency) Any test that satisfies 8.3.1 and 8.3.2 is a UMP level $\alpha$ test
- ( Necessity) if there exist a test satisfying 8.3.1 and 8.3.2 with $k > 0$, then every UMP level $\alpha$ test is a size $\alpha$ test (satisfies 8.3.2), and every UMP level $\alpha$ test satisfies 8.3.1 except perhaps on a set $A$ satisfying $\Pr(X \in A|\theta_0) = \Pr(X \in A|\theta_1) = 0$.

Example of Neyman-Pearson Lemma (cont’d)

- Suppose that $3/4 < k < 9/4$, then UMP level $\alpha$ test rejects $H_0$ if $x = 2$.

\[ \alpha = \Pr(\text{reject }| \theta = 1/2) = \Pr(x = 2| \theta = 1/2) = \frac{1}{4} \]

- If $k > 9/4$ the UMP level $\alpha$ test always not reject $H_0$, and $\alpha = 0$.

Example of Neyman-Pearson Lemma

Let $X \in \text{Binomial}(2, \theta)$, and consider testing $H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$.

Calculating the ratios of the pmfs given,

\[
\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{1}{4}, \quad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{3}{4}, \quad \frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{9}{4}
\]

- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and UMP level $\alpha$ test always rejects $H_0$. Therefore $\alpha = \Pr(\text{reject } H_0| \theta = 1/2) = 1$.

- Suppose that $1/4 < k < 3/4$, then $R = \{1, 2\}$, and UMP level $\alpha$ test rejects $H_0$ if $x = 1$ or $x = 2$.

\[
\alpha = \Pr(\text{reject }| \theta = 1/2) = \Pr(x = 1| \theta = 1/2) + \Pr(x = 2| \theta = 1/2) = \frac{3}{4}
\]

Example - Normal Distribution

$X_i \sim \text{N}(\theta, \sigma^2)$ where $\sigma^2$ is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.

\[
f(x|\theta) = \prod_{i=1}^{n} \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\}
\]

\[
\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{\exp \left\{ -\frac{\sum_{i=1}^{n} (x_i - \theta_1)^2}{2\sigma^2} \right\}}{\exp \left\{ -\frac{\sum_{i=1}^{n} (x_i - \theta_0)^2}{2\sigma^2} \right\}}
\]

\[
= \exp \left[ -\frac{\sum_{i=1}^{n} (x_i - \theta_1)^2 + \sum_{i=1}^{n} (x_i - \theta_0)^2}{2\sigma^2} \right]
\]

\[
= \exp \left[ \sum_{i=1}^{n} (x_i - \theta_1)^2 - \sum_{i=1}^{n} (x_i - \theta_0)^2 \right] \frac{2\sigma^2}{2\sigma^2}
\]

\[
= \exp \left[ n(\theta_0^2 - \theta_1^2) + 2 \sum_{i=1}^{n} x_i (\theta_1 - \theta_0) \right]
\]
Example (cont’d)

UMP level $\alpha$ test rejects if

$$\exp \left[ \frac{n(\theta_0^2 - \theta_1^2) + 2 \sum_{i=1}^n x_i(\theta_1 - \theta_0)}{2\sigma^2} \right] > k$$

$$\iff \frac{n(\theta_0^2 - \theta_1^2) + 2 \sum_{i=1}^n x_i(\theta_1 - \theta_0)}{2\sigma^2} > \log k$$

$$\iff \sum_{i=1}^n x_i > k^*$$

$$\alpha = \Pr \left( \sum_{i=1}^n X_i > k^* | \theta_0 \right)$$

$$= \Pr \left( Z > \frac{k^*/n - \theta_0}{\sigma/\sqrt{n}} \right)$$

where $Z \sim \mathcal{N}(0, 1)$.

Example (cont’d)

Under $H_0$,

$$X_i \sim \mathcal{N}(\theta_0, \sigma^2)$$

$$\bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$$

$$\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

$$\alpha = \Pr \left( \sum_{i=1}^n X_i > k^* | \theta_0 \right)$$

$$= \Pr \left( Z > \frac{k^*/n - \theta_0}{\sigma/\sqrt{n}} \right)$$

Neyman-Pearson Lemma on Sufficient Statistics

$$k^*/n - \theta_0 = z_\alpha \frac{\sigma}{\sqrt{n}}$$

$$k^* = n \left( \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right)$$

Thus, the UMP level $\alpha$ test reject if $\sum X_i > k^*$, or equivalently, reject $H_0$ if $\bar{X} > k^*/n = \theta_0 + z_\alpha \sigma / \sqrt{n}$

Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(X)$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on $T$ with rejection region $S$ is a UMP level $\alpha$ test if it satisfies

$$t \in S \quad \text{if} \quad g(t|\theta_1) > k \cdot g(t|\theta_0)$$

$$t \in S^c \quad \text{if} \quad g(t|\theta_1) < k \cdot g(t|\theta_0)$$

For some $k > 0$ and $\alpha = \Pr(T \in S|\theta_0)$
Proof

The rejection region in the sample space is

\[ R = \{ x : T(x) = t \in S \} = \{ x : g(T(x)|\theta_1) > kg(T(x)|\theta_0) \} \]

By Factorization Theorem:

\[ f(x|\theta_i) = h(x)g(T(x)|\theta_i) \]

\[ R = \{ x : g(T(x)|\theta_1)h(x) > kg(T(x)|\theta_0)h(x) \} = \{ x : f(x|\theta_1) > kf(x|\theta_0) \} \]

By Neyman-Pearson Lemma, this test is the UMP level \( \alpha \) test, and

\[ \alpha = \Pr(X \in R) = \Pr(T(X) \in S|\theta_0) \]

Revisiting the Example (cont’d)

UMP level \( \alpha \) test reject if

\[ \exp \left\{ \frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\} > k \]

\[ \iff \frac{1}{2\sigma^2/n} \left[ -\theta_0^2 + 2t(\theta_1 - \theta_0) \right] > \log k \]

\[ \iff \bar{X} = t > k^* \]

Revisiting the Example of Normal Distribution

\( X \overset{i.i.d.}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\( T = \bar{X} \) is a sufficient statistic for \( \theta \), where \( T(X) \sim \mathcal{N}(\theta, \sigma^2/n) \).

\[ g(t|\theta_i) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ -\frac{(t - \theta_i)^2}{2\sigma^2/n} \right\} \]

\[ \frac{g(t|\theta_1)}{g(t|\theta_0)} = \exp \left\{ \frac{-(t-\theta_1)^2}{2\sigma^2/n} - \frac{-(t-\theta_0)^2}{2\sigma^2/n} \right\} \]

\[ = \exp \left\{ -\frac{1}{2\sigma^2/n} \left[ (t - \theta_1)^2 - (t - \theta_0)^2 \right] \right\} \]

\[ = \exp \left\{ -\frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\} \]

Under \( H_0 \), \( \bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n) \). \( k^* \) satisfies

\[ \Pr(\text{reject } H_0|\theta_0) = \alpha \]

\[ \alpha = \Pr(\bar{X} > k^*|\theta_0) = \Pr \left( \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right) = \Pr \left( Z > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right) \]

\[ \frac{k^* - \theta_0}{\sigma/\sqrt{n}} = z_\alpha \]

\[ k^* = \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \]
Monotone Likelihood Ratio

**Definition**

A family of pdfs or pmfs \( \{ g(t|\theta) : \theta \in \Omega \} \) for a univariate random variable \( T \) with real-valued parameter \( \theta \) have a monotone likelihood ratio if \( \frac{g(t|\theta_2)}{g(t|\theta_1)} \) is an increasing (or non-decreasing) function of \( t \) for every \( \theta_2 > \theta_1 \) on \( \{ t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0 \} \).

Note: we may define MLR using decreasing function of \( t \). But all following theorems are stated according to the definition.

**Proof**

Suppose that \( \theta_2 > \theta_1 \).

\[
\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{h(t) c(\theta_2) \exp[w(\theta_2) t]}{h(t) c(\theta_1) \exp[w(\theta_1) t]} = \frac{c(\theta_2)}{c(\theta_1)} \exp[\{ w(\theta_2) - w(\theta_1) \} t]
\]

If \( w(\theta) \) is a non-decreasing function of \( \theta \), then \( w(\theta_2) - w(\theta_1) \geq 0 \) and \( \exp[\{ w(\theta_2) - w(\theta_1) \} t] \) is an increasing function of \( t \). Therefore, \( \frac{g(t|\theta_2)}{g(t|\theta_1)} \) is a non-decreasing function of \( t \), and \( T \) has MLR if \( w(\theta) \) is a non-decreasing function of \( \theta \).

Example of Monotone Likelihood Ratio

- Normal, Poisson, Binomial have the MLR Property (Exercise 8.25)
- If \( T \) is from an exponential family with the pdf or pmf
  \[
g(t|\theta) = h(t) c(\theta) \exp[w(\theta) \cdot t]
\]
  Then \( T \) has an MLR if \( w(\theta) \) is a non-decreasing function of \( \theta \).

**Karlin-Rabin Theorem**

**Theorem 8.1.17**

Suppose \( T(X) \) is a sufficient statistic for \( \theta \) and the family \( \{ g(t|\theta) : \theta \in \Omega \} \) is an MLR family. Then

1. For testing \( H_0 : \theta \leq \theta_0 \) vs \( H_1 : \theta > \theta_0 \), the UMP level \( \alpha \) test is given by rejecting \( H_0 \) if and only if \( T > t_0 \) where \( \alpha = \Pr(T > t_0|\theta_0) \).
2. For testing \( H_0 : \theta \geq \theta_0 \) vs \( H_1 : \theta < \theta_0 \), the UMP level \( \alpha \) test is given by rejecting \( H_0 \) if and only if \( T < t_0 \) where \( \alpha = \Pr(T < t_0|\theta_0) \).
Example Application of Karlin-Rabin Theorem

Let $X_i \overset{i.i.d.}{\sim} \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known, Find the UMP level $\alpha$ test for $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$.

$T(X) = \overline{X}$ is a sufficient statistic for $\theta$, and $T \sim \mathcal{N}(\theta, \sigma^2/n)$.

$$g(t|\theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ - \frac{(t-\theta)^2}{2\sigma^2/n} \right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ - \frac{\theta^2}{2\sigma^2/n} \right\}$$

Therefore, T is MLR property. Therefore $T$ is MLR property.

Finding a UMP level $\alpha$ test

By Karlin-Rabin, UMP level $\alpha$ test rejects $H_0$ iff. $T > t_0$ where

$$\alpha = \Pr(T > t_0 | \theta_0)$$

$$= \Pr \left( \frac{T - \theta_0}{\sigma/\sqrt{n}} > \frac{t_0 - \theta_0}{\sigma/\sqrt{n}} | \theta_0 \right)$$

$$= \Pr \left( \frac{Z}{\sigma/\sqrt{n}} > \frac{z_\alpha}{\sqrt{n}} \right)$$

where $Z \sim \mathcal{N}(0, 1)$.

$$\frac{t_0 - \theta_0}{\sigma/\sqrt{n}} = z_\alpha$$

$$\Rightarrow t_0 = \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$$

UMP level $\alpha$ test rejects $H_0$ if $T = \overline{X} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$.

Normal Example with Known Mean

Let $X_i \overset{i.i.d.}{\sim} \mathcal{N}(\mu_0, \sigma^2)$ where $\sigma^2$ is unknown and $\mu_0$ is known. Find the UMP level $\alpha$ test for testing $H_0: \sigma^2 \leq \sigma_0^2$ vs. $H_1: \sigma^2 > \sigma_0^2$. Let

$T = \sum_{i=1}^{n} (X_i - \mu_0)^2$ is sufficient for $\sigma^2$. To check whether $T$ has MLR property, we need to find $g(t|\sigma^2)$.

$$\frac{X_i - \mu_0}{\sigma} \sim \mathcal{N}(0, 1)$$

$$\left( \frac{X_i - \mu_0}{\sigma} \right)^2 \sim \chi_1^2$$

$$Y = T/\sigma^2 = \sum_{i=1}^{n} \left( \frac{X_i - \mu_0}{\sigma} \right)^2 \sim \chi_n^2$$

$$f_Y(y) = \frac{1}{\Gamma(\frac{n}{2})} \left( \frac{n}{2} \right)^{n/2} y^{n/2-1} e^{-y/2}$$
Normal Example with Known Mean (cont’d)

\[ f_T(t) = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \sigma^2} \left( \frac{t}{\sigma^2} \right)^{\frac{n}{2} - 1} e^{-\frac{t}{2\sigma^2}} \frac{dy}{dt} \]

\[ = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \sigma^2} \left( \frac{t}{\sigma^2} \right)^{\frac{n}{2} - 1} e^{-\frac{t}{2\sigma^2}} \frac{1}{\sigma^2} \]

\[ = \frac{t^{\frac{n}{2} - 1}}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \sigma^2} \left( \frac{1}{\sigma^2} \right)^{\frac{n}{2}} e^{-\frac{t}{2\sigma^2}} \]

\[ = h(t)c(\sigma^2) \exp[w(\sigma^2) t] \]

where \( w(\sigma^2) = -\frac{1}{2\sigma^2} \) is an increasing function in \( \sigma^2 \). Therefore, \( T = \sum_{i=1}^{n} (X_i - \mu_0)^2 \) has the MLR property.

Remarks

- For many problems, UMP level \( \alpha \) test does not exist (Example 8.3.19).
- In such cases, we can restrict our search among a subset of tests, for example, all unbiased tests.

Summary

By Karlin-Rabin Theorem, UMP level \( \alpha \) rejects \( s H_0 \) if and only if \( T > t_0 \) where \( t_0 \) is chosen such that \( \alpha = \Pr(T > t_0|\sigma_0^2) \).

Note that \( \frac{T}{\sigma_0^2} \sim \chi^2_n \)

\[ \Pr(T > t_0|\sigma_0^2) = \Pr \left( \frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2} \middle| \sigma_0^2 \right) \]

\[ = \frac{T}{\sigma_0^2} \sim \chi^2_n \]

\[ \Pr \left( \chi^2_n > \frac{t_0}{\sigma_0^2} \right) = \alpha \]

\[ = \frac{t_0}{\sigma_0^2} = \chi^2_{n,\alpha} \]

\[ t_0 = \sigma_0^2 \chi^2_{n,\alpha} \]

where \( \chi^2_{n,\alpha} \) satisfies \( \int_{\chi^2_{n,\alpha}}^{\infty} f_{\chi^2_n}(x) dx = \alpha \).

Today

- Uniformly Most Powerful Test
- Neyman-Pearson Lemma
- Monotone Likelihood Ratio
- Karlin-Rabin Theorem

Next Lecture

- Asymptotics of LRT
- Wald Test