Uniformly Most Powerful Test

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What are the typical steps for constructing a likelihood ratio test?

Is LRT statistic based on sufficient statistic identical to the LRT based on the full data?

When multiple parameters need to be estimated, what is the difference in constructing LRT?

What is unbiased test?
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• What are the typical steps for constructing a likelihood ratio test?
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• When multiple parameters need to be estimated, what is the difference in constructing LRT?
• What is unbiased test?
LRT based on sufficient statistics

**Theorem 8.2.4**

If $T(X)$ is a sufficient statistic for $\theta$, $\lambda^*(t)$ is the LRT statistic based on $T$, and $\lambda(x)$ is the LRT statistic based on $x$ then
LRT based on sufficient statistics

Theorem 8.2.4

If $T(X)$ is a sufficient statistic for $\theta$, $\lambda^*(t)$ is the LRT statistic based on $T$, and $\lambda(x)$ is the LRT statistic based on $x$ then

$$\lambda^*[T(x)] = \lambda(x)$$
LRT based on sufficient statistics

Theorem 8.2.4
If \( T(X) \) is a sufficient statistic for \( \theta \), \( \lambda^*(t) \) is the LRT statistic based on \( T \), and \( \lambda(x) \) is the LRT statistic based on \( x \) then
\[
\lambda^*[T(x)] = \lambda(x)
\]
for every \( x \) in the sample space.
Unbiased Test

**Definition**

If a test always satisfies

\[ \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is false}) \geq \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \]

Then the test is said to be unbiased.

**Alternative Definition**

Recall that \((\cdot) = \Pr(\text{reject } H_0)\). A test is unbiased if \((\cdot) = (\cdot)\) for every \(\cdot \in \Omega_0, \cdot \in \Omega_{c0}\).
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Recall that \( \beta(\theta) = \Pr(\text{reject } H_0) \). A test is unbiased if
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Alternative Definition
Recall that \( \beta(\theta) = \Pr(\text{reject } H_0) \). A test is unbiased if
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for every \( \theta' \in \Omega^c_0 \) and \( \theta \in \Omega_0 \).
Example

\(X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)\) where \(\sigma^2\) is known, testing \(H_0 : \theta \leq \theta_0\) vs \(H_1 : \theta > \theta_0\).
Example

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LRT test rejects \( H_0 \) if

\[ \frac{\bar{x} - \theta_0}{\sigma / \sqrt{n}} > c. \]
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\beta(\theta) = \Pr \left( \frac{\bar{X} - \theta_0}{\sigma / \sqrt{n}} > c \right)
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\beta(\theta) = \Pr \left( \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c \right) \\
= \Pr \left( \frac{\bar{X} - \theta + \theta - \theta_0}{\sigma/\sqrt{n}} > c \right)
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= \Pr \left( \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} + \frac{\theta - \theta_0}{\sigma/\sqrt{n}} > c \right)
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\(X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)\) where \(\sigma^2\) is known, testing \(H_0 : \theta \leq \theta_0\) vs \(H_1 : \theta > \theta_0\).

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Example

$X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known, testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

LRT test rejects $H_0$ if \[
\frac{\bar{x} - \theta_0}{\sigma / \sqrt{n}} > c.
\]

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\beta(\theta) = \Pr \left( \frac{\bar{X} - \theta_0}{\sigma / \sqrt{n}} > c \right) = \Pr \left( \frac{\bar{X} - \theta + \theta - \theta_0}{\sigma / \sqrt{n}} > c \right) = \Pr \left( \frac{\bar{X} - \theta}{\sigma / \sqrt{n}} + \frac{\theta - \theta_0}{\sigma / \sqrt{n}} > c \right) = \Pr \left( \frac{\bar{X} - \theta}{\sigma / \sqrt{n}} > c - \frac{\theta - \theta_0}{\sigma / \sqrt{n}} \right)
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Note that $X_i \sim \mathcal{N}(\theta, \sigma^2)$, $\bar{X} \sim \mathcal{N}(\theta, \sigma^2 / n)$, and $\frac{\bar{X} - \theta}{\sigma / \sqrt{n}} \sim \mathcal{N}(0, 1)$. 
Example (cont’d)

Therefore, for $Z \sim \mathcal{N}(0, 1)$

$$
\beta(\theta) = \Pr \left( Z > c + \frac{\theta_0 - \theta}{\sigma / \sqrt{n}} \right)
$$
Example (cont’d)

Therefore, for \( Z \sim \mathcal{N}(0, 1) \)

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\beta(\theta) = \Pr \left( Z > c + \frac{\theta_0 - \theta}{\sigma / \sqrt{n}} \right)
\]

Because the power function is increasing function of \( \theta \),

\[
\beta(\theta') \geq \beta(\theta)
\]
Example (cont’d)

Therefore, for $Z \sim \mathcal{N}(0, 1)$

$$\beta(\theta) = \Pr \left( Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right)$$

Because the power function is increasing function of $\theta$, 

$$\beta(\theta') \geq \beta(\theta)$$

always holds when $\theta \leq \theta_0 < \theta'$. Therefore the LRTs are unbiased.
Uniformly Most Powerful Test (UMP)

Definition

Let \( C \) be a class of tests between \( H_0 : \theta \in \Omega \) vs \( H_1 : \theta \in \Omega_0^c \). A test in \( C \), with power function \( \beta(\theta) \) is uniformly most powerful (UMP) test in class \( C \) if \( \beta(\theta) \geq \beta'(\theta) \) for every \( \theta \in \Omega_0^c \) and every \( \beta'(\theta) \), which is a power function of another test in \( C \).
Consider $\mathcal{C}$ be the set of all the level $\alpha$ test. The UMP test in this class is called a UMP level $\alpha$ test.
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UMP level $\alpha$ test has the smallest type II error probability for any $\theta \in \Omega_0^c$ in this class.
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- A UMP test is "uniform" in the sense that it is most powerful for every $\theta \in \Omega^c_0$. 
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- A UMP test is "uniform" in the sense that it is most powerful for every $\theta \in \Omega_c^0$.
- For simple hypothesis such as $H_0: \theta = \theta_0$ and $H_1: \theta = \theta_1$, UMP level $\alpha$ test always exists.
Theorem 8.3.12 - Neyman-Pearson Lemma

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding the $\theta_i$ is $f(x|\theta_i)$, $i = 0, 1$, using a test with rejection region $R$ that satisfies

For some $k > 0$ and $\beta = \Pr(X^2 \in R | \theta = \theta_0)$, Then,

- (Sufficiency) Any test that satisfies 8.3.1 and 8.3.2 is a UMP level test
- (Necessity) if there exist a test satisfying 8.3.1 and 8.3.2 with $k > 0$, then every UMP level test is a size test (satisfies 8.3.2), and every UMP level test satisfies 8.3.1 except perhaps on a set $A$ satisfying $\Pr(X^2 \in A | \theta = \theta_0) = \Pr(X^2 \in A | \theta = \theta_1) = 0$. 
Neyman-Pearson Lemma

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\[
x \in R \quad \text{if} \quad f(x|\theta_1) > kf(x|\theta_0)
\]  \hspace{1cm} (8.3.1) and

For some \( k > 0 \) and \( \alpha = \Pr(X \in R| \theta_0) \), Then,

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\[
\begin{align*}
x \in R & \quad \text{if } f(x|\theta_1) > kf(x|\theta_0) \\
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(8.3.1) and (8.3.2)
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For some $k \geq 0$ and $\alpha = \Pr(X \in R|\theta_0)$, Then,

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Example of Neyman-Pearson Lemma

Let $X \in \text{Binomial}(2, \theta)$, and consider testing

$H_0: \theta = 0.5$ vs. $H_1: \theta = 0.75$.

Calculating the ratios of the pmfs given,

$f(0 \mid 1) / f(0 \mid 0) = 1/4$;

$f(1 \mid 1) / f(1 \mid 0) = 3/4$;

$f(2 \mid 1) / f(2 \mid 0) = 9/4$.

Suppose that $k < 1/4$, then the rejection region $R = \{0; 1; 2\}$, and UMP level test always rejects $H_0$. Therefore $\Pr(\text{reject} \mid \theta = 0.5) = 1$.

Suppose that $1/4 < k < 3/4$, then $R = \{1; 2\}$, and UMP level test rejects $H_0$ if $x = 1$ or $x = 2$. Therefore $\Pr(\text{reject} \mid \theta = 0.5) = \Pr(x = 1 \mid \theta = 0.5) + \Pr(x = 2 \mid \theta = 0.5) = 3/4$. 
Example of Neyman-Pearson Lemma

Let $X \sim \text{Binomial}(2, \theta)$, and consider testing
$H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$. 
Example of Neyman-Pearson Lemma

Let $X \sim \text{Binomial}(2, \theta)$, and consider testing $H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$. Calculating the ratios of the pmfs given,

\[
\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{1}{4}, \quad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{3}{4}, \quad \frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{9}{4}
\]
Example of Neyman-Pearson Lemma

Let $X \sim \text{Binomial}(2, \theta)$, and consider testing

$H_0 : \theta = \theta_0 = 1/2 \text{ vs. } H_1 : \theta = \theta_1 = 3/4$.

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- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and UMP level $\alpha$ test always rejects $H_0$. Therefore

$\alpha = \Pr(\text{reject } H_0|\theta = \theta_0 = 1/2) = 1$. 

Example of Neyman-Pearson Lemma

Let $X \sim \text{Binomial}(2, \theta)$, and consider testing
$H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$.

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- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and
  UMP level $\alpha$ test always rejects $H_0$. Therefore
  $\alpha = \Pr(\text{reject } H_0|\theta = \theta_0 = 1/2) = 1$.

- Suppose that $1/4 < k < 3/4$, then $R = \{1, 2\}$, and UMP level $\alpha$ test
  rejects $H_0$ if $x = 1$ or $x = 2$. 
Example of Neyman-Pearson Lemma

Let $X \in \text{Binomial}(2, \theta)$, and consider testing

$H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$.

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\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{1}{4}, \quad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{3}{4}, \quad \frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{9}{4}.
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- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and UMP level $\alpha$ test always rejects $H_0$. Therefore

  $\alpha = \Pr(\text{reject } H_0|\theta = \theta_0 = 1/2) = 1$.

- Suppose that $1/4 < k < 3/4$, then $R = \{1, 2\}$, and UMP level $\alpha$ test rejects $H_0$ if $x = 1$ or $x = 2$.

  $\alpha = \Pr(\text{reject }|\theta = 1/2) = \Pr(x = 1|\theta = 1/2) + \Pr(x = 2|\theta = 1/2) = \frac{3}{4}$.
Example of Neyman-Pearson Lemma (cont’d)

- Suppose that $3/4 < k < 9/4$, then UMP level $\alpha$ test rejects $H_0$ if $x = 2$
Example of Neyman-Pearson Lemma (cont’d)

- Suppose that $3/4 < k < 9/4$, then UMP level $\alpha$ test rejects $H_0$ if $x = 2$

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Example of Neyman-Pearson Lemma (cont’d)

- Suppose that $3/4 < k < 9/4$, then UMP level $\alpha$ test rejects $H_0$ if $x = 2$

\[
\alpha = \Pr(\text{reject} | \theta = 1/2) = \Pr(x = 2 | \theta = 1/2) = \frac{1}{4}
\]

- If $k > 9/4$ the UMP level $\alpha$ test always not reject $H_0$, and $\alpha = 0$
Example - Normal Distribution

\( X_i \overset{i.i.d.}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).
Example - Normal Distribution

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\[
f(x|\theta) = \prod_{i=1}^{n} \left[ \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\} \right]
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Example - Normal Distribution

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\[
f(x|\theta_1) = \exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right\}
\]

\[
\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{\exp \left\{ -\sum_{i=1}^{n}(x_i - \theta_1)^2 \right\}}{\exp \left\{ -\sum_{i=1}^{n}(x_i - \theta_0)^2 \right\}}
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\( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\[
\begin{align*}
    f(x|\theta) &= \prod_{i=1}^{n} \left[ \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\} \right] \\
    \frac{f(x|\theta_1)}{f(x|\theta_0)} &= \frac{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right\}}{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right\}} \\
    &= \exp \left[ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right]
\end{align*}
\]
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$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.

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f(x|\theta) = \prod_{i=1}^{n} \left[ \frac{1}{2\pi \sigma^2} \exp \left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\} \right]
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\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right\}}{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right\}}
\]

\[
= \exp \left[ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right]
\]

\[
= \exp \left[ \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right]
\]
Example - Normal Distribution

$X_i \overset{i.i.d.}{\sim} \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.

$$f(x|\theta) = \prod_{i=1}^{n} \left[ \frac{1}{2\pi\sigma^2} \exp \left\{ -\frac{(x_i - \theta)^2}{2\sigma^2} \right\} \right]$$

$$\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right\}}{\exp \left\{ -\frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right\}}$$

$$= \exp \left[ -\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2}{2\sigma^2} \right]$$

$$= \exp \left[ \frac{\sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \theta_1)^2}{2\sigma^2} \right]$$

$$= \exp \left[ \frac{n(\theta_0^2 - \theta_1^2) + 2\sum_{i=1}^{n}x_i(\theta_1 - \theta_0)}{2\sigma^2} \right]$$
Example (cont’d)

UMP level $\alpha$ test rejects if

$$\exp \left[ \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} \right] > k$$
Example (cont’d)

UMP level $\alpha$ test rejects if

$$\exp \left[ \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} \right] > k$$

$$\iff \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} > \log k$$
Example (cont’d)

UMP level $\alpha$ test rejects if

$$\exp \left[ \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} \right] > k$$

$$\iff \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} > \log k$$

$$\iff \sum_{i=1}^{n} x_i > k^*$$
UMP level $\alpha$ test rejects if

$$\exp \left[ \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} \right] > k$$

$$\Leftrightarrow \frac{n(\theta_0^2 - \theta_1)^2 + 2 \sum_{i=1}^{n} x_i(\theta_1 - \theta_0)}{2\sigma^2} > \log k$$

$$\Leftrightarrow \sum_{i=1}^{n} x_i > k^*$$

$$\alpha = \Pr \left( \sum_{i=1}^{n} X_i > k^* | \theta_0 \right)$$
Example (cont’d)

Under $H_0$,

$$X_i \sim \mathcal{N}(\theta_0, \sigma^2)$$
Example (cont’d)

Under $H_0$,

$$X_i \sim \mathcal{N}(\theta_0, \sigma^2)$$

$$\bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$$
Example (cont’d)

Under $H_0$,

\[ X_i \sim \mathcal{N}(\theta_0, \sigma^2) \]
\[ \bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n) \]
\[ \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1) \]
Example (cont’d)

Under $H_0$,

\[
X_i \sim \mathcal{N}(\theta_0, \sigma^2) \\
\bar{X} \sim \mathcal{N}(\theta_0, \sigma^2 / n) \\
\frac{\bar{X} - \theta_0}{\sigma / \sqrt{n}} \sim \mathcal{N}(0, 1)
\]

\[
\alpha = \Pr \left( \sum_{i=1}^{n} X_i > k^* | \theta_0 \right) \\
= \Pr \left( Z > \frac{k^* / n - \theta_0}{\sigma / \sqrt{n}} \right)
\]

where $Z \sim \mathcal{N}(0, 1)$. 

Example (cont’d)

\[
\frac{k^* / n - \theta_0}{\sigma / \sqrt{n}} = z_\alpha
\]
Example (cont’d)

\[ \frac{k^*/n - \theta_0}{\sigma/\sqrt{n}} = z_\alpha \]

\[ k^* = n \left( \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right) \]
Example (cont’d)

\[
\frac{k^*/n - \theta_0}{\sigma/\sqrt{n}} = z_\alpha
\]

\[k^* = n \left( \theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right)\]

Thus, the UMP level \( \alpha \) test reject if \( \sum X_i > k^* \), or equivalently, reject \( H_0 \) if \( \bar{X} > k^*/n = \theta_0 + z_\alpha \sigma/\sqrt{n} \)
Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(X)$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on $T$ with rejection region $S$ is a UMP level $\alpha$ test if it satisfies
Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(X)$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on $T$ with rejection region $S$ is a UMP level $\alpha$ test if it satisfies

$$t \in S \text{ if } g(t|\theta_1) > k \cdot g(t|\theta_0)$$

and

$$t \not\in S \text{ if } g(t|\theta_1) < k \cdot g(t|\theta_0)$$
Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(X)$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on $T$ with rejection region $S$ is a UMP level $\alpha$ test if it satisfies

$$
t \in S \quad \text{if} \quad g(t|\theta_1) > k \cdot g(t|\theta_0) \quad \text{and}
$$
$$
t \in S^c \quad \text{if} \quad g(t|\theta_1) < k \cdot g(t|\theta_0)
$$

For some $k > 0$ and $\alpha = \Pr(T \in S | \theta_0)$.
Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(X)$ is a sufficient statistic for $\theta$ and $g(t|\theta_i)$ is the pdf or pmf of $T$. Corresponding $\theta_i, i \in \{0, 1\}$. Then any test based on $T$ with rejection region $S$ is a UMP level $\alpha$ test if it satisfies

\[
\begin{align*}
  t &\in S & \text{if } g(t|\theta_1) > k \cdot g(t|\theta_0) \text{ and } \\
  t &\in S^c & \text{if } g(t|\theta_1) < k \cdot g(t|\theta_0)
\end{align*}
\]

For some $k > 0$ and $\alpha = \Pr(T \in S|\theta_0)$
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]

\[ = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) > k g(T(\mathbf{x})|\theta_0) \} \]
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]

\[ = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) > kg(T(\mathbf{x})|\theta_0) \} \]

By Factorization Theorem:

\[ f(\mathbf{x}|\theta_i) = h(\mathbf{x})g(T(\mathbf{x})|\theta_i) \]
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]
\[ = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) > k g(T(\mathbf{x})|\theta_0) \} \]

By Factorization Theorem:

\[ f(\mathbf{x}|\theta_i) = h(\mathbf{x}) g(T(\mathbf{x})|\theta_i) \]
\[ R = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) h(x) > k g(T(\mathbf{x})|\theta_0) h(x) \} \]
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]
\[ = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) > k g(T(\mathbf{x})|\theta_0) \} \]

By Factorization Theorem:

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\[ = \{ \mathbf{x} : f(\mathbf{x}|\theta_1) > k f(\mathbf{x}|\theta_0) \} \]
Proof

The rejection region in the sample space is

\[ R = \{ \mathbf{x} : T(\mathbf{x}) = t \in S \} \]
\[ = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1) > kg(T(\mathbf{x})|\theta_0) \} \]

By Factorization Theorem:

\[ f(\mathbf{x}|\theta_i) = h(\mathbf{x})g(T(\mathbf{x})|\theta_i) \]
\[ R = \{ \mathbf{x} : g(T(\mathbf{x})|\theta_1)h(x) > kg(T(\mathbf{x})|\theta_0)h(x) \} \]
\[ = \{ \mathbf{x} : f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0) \} \]

By Neyman-Pearson Lemma, this test is the UMP level \( \alpha \) test, and

\[ \alpha = \Pr(\mathbf{X} \in R) = \Pr(T(\mathbf{X}) \in S|\theta_0) \]
Revisiting the Example of Normal Distribution

\[ X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \] where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).
Revisiting the Example of Normal Distribution

\[ X_i \sim \text{i.i.d. } N(\theta, \sigma^2) \] where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\( T = \bar{X} \) is a sufficient statistic for \( \theta \), where \( T \sim N(\theta, \sigma^2/n) \).
Revisiting the Example of Normal Distribution

\( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\( T = \bar{X} \) is a sufficient statistic for \( \theta \), where \( T \sim \mathcal{N}(\theta, \sigma^2/n) \).

\[
g(t|\theta_i) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ -\frac{(t - \theta_i)^2}{2\sigma^2/n} \right\}
\]
Revisiting the Example of Normal Distribution

\(X_i \sim_{i.i.d.} \mathcal{N}(\theta, \sigma^2)\) where \(\sigma^2\) is known. Consider testing \(H_0 : \theta = \theta_0\) vs. \(H_1 : \theta = \theta_1\) where \(\theta_1 > \theta_0\).

\(T = \bar{X}\) is a sufficient statistic for \(\theta\), where \(T \sim \mathcal{N}(\theta, \sigma^2/n)\).

\[
g(t|\theta_i) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ -\frac{(t - \theta_i)^2}{2\sigma^2/n} \right\}
\]

\[
g(t|\theta_1) = \frac{\exp \left\{ -\frac{(t - \theta_1)^2}{2\sigma^2/n} \right\}}{\exp \left\{ -\frac{(t - \theta_0)^2}{2\sigma^2/n} \right\}}
\]
Revisiting the Example of Normal Distribution

\( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\( T = \overline{X} \) is a sufficient statistic for \( \theta \), where \( T \sim \mathcal{N}(\theta, \sigma^2/n) \).

\[
\begin{align*}
g(t|\theta_i) &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ -\frac{(t - \theta_i)^2}{2\sigma^2/n} \right\} \\
g(t|\theta_1) &= \exp \left\{ -\frac{(t - \theta_1)^2}{2\sigma^2/n} \right\} \\
g(t|\theta_0) &= \exp \left\{ -\frac{(t - \theta_0)^2}{2\sigma^2/n} \right\} \\
n &= \exp \left\{ -\frac{1}{2\sigma^2/n} \left[ (t - \theta_1)^2 - (t - \theta_0)^2 \right] \right\}
\end{align*}
\]
Revisiting the Example of Normal Distribution

\( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known. Consider testing \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta = \theta_1 \) where \( \theta_1 > \theta_0 \).

\( T = \overline{X} \) is a sufficient statistic for \( \theta \), where \( T \sim \mathcal{N}(\theta, \sigma^2/n) \).

\[
\begin{align*}
g(t|\theta_i) &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{ -\frac{(t - \theta_i)^2}{2\sigma^2/n} \right\} \\
g(t|\theta_1) &= \frac{\exp\left\{ -\frac{(t - \theta_1)^2}{2\sigma^2/n} \right\}}{\exp\left\{ -\frac{(t - \theta_0)^2}{2\sigma^2/n} \right\}} \\
&= \exp\left\{ -\frac{1}{2\sigma^2/n} \left[ (t - \theta_1)^2 - (t - \theta_0)^2 \right] \right\} \\
&= \exp\left\{ -\frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\}
\end{align*}
\]
Revisiting the Example (cont’d)

UMP level $\alpha$ test reject if

$$\exp \left\{ -\frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\} > k$$
Revisiting the Example (cont’d)

UMP level \( \alpha \) test reject if

\[
\exp \left\{ -\frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\} > k
\]

\[\iff \frac{1}{2\sigma^2/n} \left[ -\left( \theta_1^2 - \theta_0^2 \right) + 2t(\theta_1 - \theta_0) \right] > \log k\]
Revisiting the Example (cont’d)

UMP level $\alpha$ test reject if

$$
\exp \left\{ -\frac{1}{2\sigma^2/n} \left[ \theta_1^2 - \theta_0^2 - 2t(\theta_1 - \theta_0) \right] \right\} > k
$$

$$
\iff \quad \frac{1}{2\sigma^2/n} \left[ -(\theta_1^2 - \theta_0^2) + 2t(\theta_1 - \theta_0) \right] > \log k
$$

$$
\iff \quad \bar{X} = t > k^*
$$
Under $H_0$, $\bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. $k^*$ satisfies
Revisiting the Example (cont’d)

Under $H_0$, $\bar{X} \sim N(\theta_0, \sigma^2/n)$. $k^*$ satisfies

$$\Pr(\text{reject } H_0 | \theta_0) = \alpha$$
Revisiting the Example (cont’d)

Under $H_0$, $\bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. $k^*$ satisfies

$$\Pr(\text{reject } H_0|\theta_0) = \alpha$$

$$\alpha = \Pr(\bar{X} > k^*|\theta_0)$$
Under $H_0$, $\bar{X} \sim N(\theta_0, \sigma^2/n)$. $k^*$ satisfies

$$
\Pr(\text{reject } H_0 | \theta_0) = \alpha \\
\alpha = \Pr(\bar{X} > k^* | \theta_0) \\
= \Pr \left( \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right)
$$
Revisiting the Example (cont’d)

Under \( H_0, \bar{X} \sim \mathcal{N}(\theta_0, \sigma^2/n) \). \( k^* \) satisfies

\[
\begin{align*}
\Pr(\text{reject } H_0|\theta_0) &= \alpha \\
\alpha &= \Pr(\bar{X} > k^*|\theta_0) \\
&= \Pr \left( \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right) \\
&= \Pr \left( Z > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right)
\end{align*}
\]
Under $H_0$, $\overline{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. $k^*$ satisfies

$$\Pr(\text{reject } H_0 | \theta_0) = \alpha$$

$$\alpha = \Pr(\overline{X} > k^* | \theta_0)$$

$$= \Pr \left( \frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right)$$

$$= \Pr \left( Z > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right)$$

$$\frac{k^* - \theta_0}{\sigma/\sqrt{n}} = z_\alpha$$
Revisiting the Example (cont’d)

Under $H_0$, $\overline{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. $k^*$ satisfies

$$
\Pr(\text{reject } H_0 | \theta_0) = \alpha \\
\alpha = \Pr(\overline{X} > k^* | \theta_0) \\
= \Pr \left( \frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right) \\
= \Pr \left( Z > \frac{k^* - \theta_0}{\sigma/\sqrt{n}} \right) \\
\frac{k^* - \theta_0}{\sigma/\sqrt{n}} = z_\alpha \\
k^* = \theta_0 + z_\alpha \frac{\sigma}{n}$$
Monotone Likelihood Ratio

**Definition**

A family of pdfs or pmfs \( \{g(t|\theta) : \theta \in \Omega\} \) for a univariate random variable \( T \) with real-valued parameter \( \theta \) have a monotone likelihood ratio if

\[
\frac{g(t|\theta_2)}{g(t|\theta_1)}
\]

is an increasing (or non-decreasing) function of \( t \) for every \( \theta_2 > \theta_1 \) on \( \{t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\} \).
Monotone Likelihood Ratio

**Definition**

A family of pdfs or pmfs \( \{ g(t|\theta) : \theta \in \Omega \} \) for a univariate random variable \( T \) with real-valued parameter \( \theta \) have a monotone likelihood ratio if

\[
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\]

is an increasing (or non-decreasing) function of \( t \) for every \( \theta_2 > \theta_1 \) on \( \{ t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0 \} \).

Note: we may define MLR using decreasing function of \( t \). But all following theorems are stated according to the definition.
Example of Monotone Likelihood Ratio

- Normal, Poisson, Binomial have the MLR Property (Exercise 8.25)
Example of Monotone Likelihood Ratio

- Normal, Poisson, Binomial have the MLR Property (Exercise 8.25)
- If $T$ is from an exponential family with the pdf or pmf

$$g(t|\theta) = h(t)c(\theta) \exp[w(\theta) \cdot t]$$

Then $T$ has an MLR if $w(\theta)$ is a non-decreasing function of $\theta$. 
Proof

Suppose that $\theta_2 > \theta_1$. 
Proof

Suppose that $\theta_2 > \theta_1$.

\[
\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{h(t)c(\theta_2) \exp[w(\theta_2)t]}{h(t)c(\theta_1) \exp[w(\theta_1)t]}
\]
Proof

Suppose that $\theta_2 > \theta_1$.

\[
\begin{align*}
\frac{g(t|\theta_2)}{g(t|\theta_1)} &= \frac{h(t) c(\theta_2) \exp[w(\theta_2) t]}{h(t) c(\theta_1) \exp[w(\theta_1) t]} \\
&= \frac{c(\theta_2)}{c(\theta_1)} \exp[\{w(\theta_2) - w(\theta_1)\} t]
\end{align*}
\]
Proof

Suppose that $\theta_2 > \theta_1$.

$$\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{h(t) c(\theta_2) \exp[w(\theta_2)t]}{h(t) c(\theta_1) \exp[w(\theta_1)t]}$$

$$= \frac{c(\theta_2)}{c(\theta_1)} \exp[\{w(\theta_2) - w(\theta_1)\}t]$$

If $w(\theta)$ is a non-decreasing function of $\theta$, then $w(\theta_2) - w(\theta_1) \geq 0$ and
Proof

Suppose that \( \theta_2 > \theta_1 \).

\[
\frac{g(t|\theta_2)}{g(t|\theta_1)} = \frac{h(t) c(\theta_2) \exp[w(\theta_2) t]}{h(t) c(\theta_1) \exp[w(\theta_1) t]}
= \frac{c(\theta_2)}{c(\theta_1)} \exp[\{w(\theta_2) - w(\theta_1)\} t]
\]

If \( w(\theta) \) is a non-decreasing function of \( \theta \), then \( w(\theta_2) - w(\theta_1) \geq 0 \) and \( \exp[\{w(\theta_2) - w(\theta_1)\} t] \) is an increasing function of \( t \). Therefore, \( \frac{g(t|\theta_2)}{g(t|\theta_1)} \) is a non-decreasing function of \( t \), and \( T \) has MLR if \( w(\theta) \) is a non-decreasing function of \( \theta \).
Theorem 8.1.17

Suppose $T(X)$ is a sufficient statistic for $\theta$ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then
Karlin-Rabin Theorem

Theorem 8.1.17

Suppose $T(X)$ is a sufficient statistic for $\theta$ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then

1. For testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, the UMP level $\alpha$ test is given by rejecting $H_0$ if and only if $T > t_0$ where $\alpha = \Pr(T > t_0|\theta_0)$. 

Hyun Min Kang

Biostatistics 602 - Lecture 20

March 28th, 2013
Karlin-Rabin Theorem

Theorem 8.1.17

Suppose $T(\mathbf{X})$ is a sufficient statistic for $\theta$ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then

1. For testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, the UMP level $\alpha$ test is given by rejecting $H_0$ if and only if $T > t_0$ where $\alpha = \Pr(T > t_0|\theta_0)$.

2. For testing $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$, the UMP level $\alpha$ test is given by rejecting $H_0$ if and only if $T < t_0$ where $\alpha = \Pr(T < t_0|\theta_0)$. 

Example Application of Karlin-Rabin Theorem

Let $X_i \sim \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known, Find the UMP level $\alpha$ test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$. 

$T(X) = \overline{X}$ is a sufficient statistic for $\theta$, and $T \sim \mathcal{N}(\theta, \sigma^2/n)$. 

$$g(t_j) = \frac{1}{\sqrt{2\pi \sigma^2/n}} \exp \left\{ \frac{(t_j - \theta)^2}{2\sigma^2/n} \right\} \frac{1}{\sqrt{2\pi \sigma^2/n}} \exp \left\{ \frac{t_j^2}{2\sigma^2/n} \right\} \exp \left\{ \frac{\theta^2}{2\sigma^2/n} \right\} = h(t_j)c(\theta) \exp [w(\theta)t_j]$$

where $w(\theta) = \frac{\theta^2}{2\sigma^2/n}$ is an increasing function in $\theta$. Therefore $T$ is MLR property.
Example Application of Karlin-Rabin Theorem

Let $X_i \overset{i.i.d.}{\sim} \mathcal{N}(\theta, \sigma^2)$ where $\sigma^2$ is known, Find the UMP level $\alpha$ test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

$T(X) = \bar{X}$ is a sufficient statistic for $\theta$, and $T \sim \mathcal{N}(\theta, \sigma^2/n)$. 

$\sqrt{2 \sigma^2 / n} \exp\left\{ -\frac{1}{2} \frac{\theta^2}{\sigma^2} \right\} = h(t) c \left( \frac{\theta}{\sigma} \right) \exp\left[ w(t) \frac{\theta}{\sigma} \right]$ where $w(t) = \frac{\theta}{\sigma}$ is an increasing function in $\theta$. Therefore $T$ is MLR property.
Example Application of Karlin-Rabin Theorem

Let \( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2) \) where \( \sigma^2 \) is known, Find the UMP level \( \alpha \) test for \( H_0 : \theta \leq \theta_0 \) vs \( H_1 : \theta > \theta_0 \).

\[ T(X) = \bar{X} \] is a sufficient statistic for \( \theta \), and \( T \sim \mathcal{N}(\theta, \sigma^2/n) \).

\[ g(t|\theta) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp \left\{ -\frac{(t - \theta)^2}{2\sigma^2/n} \right\} \]
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$$= h(t)c(\theta) \exp[w(\theta)t]$$

where $w(\theta) = \frac{\theta}{\sigma^2/n}$ is an increasing function in $\theta$. Therefore $T$ is MLR property.
Finding a UMP level $\alpha$ test

By Karlin-Rabin, UMP level $\alpha$ test rejects $H_0$ iff. $T > t_0$ where

$$\alpha = \Pr(T > t_0 | \theta_0)$$
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$$\frac{t_0 - \theta_0}{\sigma / \sqrt{n}} = z_\alpha$$
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UMP level $\alpha$ test rejects $H_0$ if $T = \bar{X} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$. 
Testing $H_0 : \theta \geq \theta_0$ vs. $H_1 : \theta < \theta_0$

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UMP level $\alpha$ test rejects $H_0$ if $T < t_0$ where

$$\alpha = \Pr(T < t_0|\theta_0) = \Pr\left(\frac{T - \theta_0}{\sigma/\sqrt{n}} < \frac{t_0 - \theta_0}{\sigma/\sqrt{n}} \bigg| \theta_0\right)$$
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\]

\[
1 - \alpha = \Pr\left(Z \geq \frac{t_0 - \theta_0}{\sigma/\sqrt{n}}\right)
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$$

$$
= \Pr\left(Z < \frac{t_0 - \theta_0}{\sigma/\sqrt{n}}\right)
$$

$$
1 - \alpha = \Pr\left(Z \geq \frac{t_0 - \theta_0}{\sigma/\sqrt{n}}\right)
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$$
\frac{t_0 - \theta_0}{\sigma/\sqrt{n}} = z_{1-\alpha}
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Testing $H_0 : \theta \geq \theta_0$ vs. $H_1 : \theta < \theta_0$

UMP level $\alpha$ test rejects $H_0$ if $T < t_0$ where

$$\alpha = \Pr(T < t_0 | \theta_0) = \Pr\left(\frac{T - \theta_0}{\sigma/\sqrt{n}} < \frac{t_0 - \theta_0}{\sigma/\sqrt{n}} \bigg| \theta_0\right)$$

$$= \Pr\left(Z < \frac{t_0 - \theta_0}{\sigma/\sqrt{n}}\right)$$

$$1 - \alpha = \Pr\left(Z \geq \frac{t_0 - \theta_0}{\sigma/\sqrt{n}}\right)$$

$$\frac{t_0 - \theta_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$$

$$t_0 = \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} = \theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha}$$

Therefore, the test rejects $H_0$ if $T < t_0 = \theta - \frac{\sigma}{\sqrt{n}} z_{\alpha}$
Normal Example with Known Mean

\( X_i \text{i.i.d. } \mathcal{N}(\mu_0, \sigma^2) \) where \( \sigma^2 \) is unknown and \( \mu_0 \) is known. Find the UMP level \( \alpha \) test for testing \( H_0 : \sigma^2 \leq \sigma_0^2 \) vs. \( H_1 : \sigma^2 > \sigma_0^2 \). Let \( T = \sum_{i=1}^{n}(X_i - \mu_0)^2 \) is sufficient for \( \sigma^2 \).
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Normal Example with Known Mean

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_0, \sigma^2)$ where $\sigma^2$ is unknown and $\mu_0$ is known. Find the UMP level $\alpha$ test for testing $H_0 : \sigma^2 \leq \sigma_0^2$ vs. $H_1 : \sigma^2 > \sigma_0^2$. Let $T = \sum_{i=1}^{n} (X_i - \mu_0)^2$ is sufficient for $\sigma^2$. To check whether $T$ has MLR property, we need to find $g(t|\sigma^2)$.

$$\frac{X_i - \mu_0}{\sigma} \sim \mathcal{N}(0, 1)$$
Normal Example with Known Mean

\( X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_0, \sigma^2) \) where \( \sigma^2 \) is unknown and \( \mu_0 \) is known. Find the UMP level \( \alpha \) test for testing \( H_0 : \sigma^2 \leq \sigma_0^2 \) vs. \( H_1 : \sigma^2 > \sigma_0^2 \). Let \( T = \sum_{i=1}^{n} (X_i - \mu_0)^2 \) is sufficient for \( \sigma^2 \). To check whether \( T \) has MLR property, we need to find \( g(t|\sigma^2) \).

\[
\begin{align*}
\frac{X_i - \mu_0}{\sigma} & \sim \mathcal{N}(0, 1) \\
\left( \frac{X_i - \mu_0}{\sigma} \right)^2 & \sim \chi_1^2
\end{align*}
\]
Normal Example with Known Mean

\( X_i \sim \text{i.i.d. } \mathcal{N}(\mu_0, \sigma^2) \) where \( \sigma^2 \) is unknown and \( \mu_0 \) is known. Find the UMP level \( \alpha \) test for testing \( H_0 : \sigma^2 \leq \sigma_0^2 \) vs. \( H_1 : \sigma^2 > \sigma_0^2 \). Let \( T = \sum_{i=1}^{n} (X_i - \mu_0)^2 \) is sufficient for \( \sigma^2 \). To check whether \( T \) has MLR property, we need to find \( g(t|\sigma^2) \).

\[
\frac{X_i - \mu_0}{\sigma} \sim \mathcal{N}(0, 1)
\]
\[
\left(\frac{X_i - \mu_0}{\sigma}\right)^2 \sim \chi_1^2
\]

\[
Y = \frac{T}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \mu_0}{\sigma}\right)^2 \sim \chi_n^2
\]
Normal Example with Known Mean

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_0, \sigma^2)$ where $\sigma^2$ is unknown and $\mu_0$ is known. Find the UMP level $\alpha$ test for testing $H_0 : \sigma^2 \leq \sigma^2_0$ vs. $H_1 : \sigma^2 > \sigma^2_0$. Let $T = \sum_{i=1}^{n} (X_i - \mu_0)^2$ is sufficient for $\sigma^2$. To check whether $T$ has MLR property, we need to find $g(t|\sigma^2)$.

\[
\frac{X_i - \mu_0}{\sigma} \sim \mathcal{N}(0, 1)
\]

\[
\left(\frac{X_i - \mu_0}{\sigma}\right)^2 \sim \chi_1^2
\]

\[
Y = T/\sigma^2 = \sum_{i=1}^{n} \left(\frac{X_i - \mu_0}{\sigma}\right)^2 \sim \chi_n^2
\]

\[
f_Y(y) = \frac{1}{\Gamma\left(\frac{n}{2}\right)2^{n/2}} y^{n/2-1} e^{-\frac{y}{2}}
\]
Normal Example with Known Mean (cont’d)

\[ f_T(t) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{t}{\sigma^2}\right)^{n/2 - 1} e^{-\frac{t}{2\sigma^2}} \left| \frac{dy}{dt} \right| \]
Normal Example with Known Mean (cont’d)

\[ f_T(t) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{t}{\sigma^2}\right)^{n/2 - 1} e^{-\frac{t}{2\sigma^2}} \left| \frac{dy}{dt} \right| \]

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Normal Example with Known Mean (cont’d)

\[ f_T(t) = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \left( \frac{t}{\sigma^2} \right)^{n/2 - 1}} e^{-\frac{t}{2\sigma^2}} \left| \frac{dy}{dt} \right| \]

\[ = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \left( \frac{t}{\sigma^2} \right)^{n/2 - 1}} e^{-\frac{t}{2\sigma^2}} \frac{1}{\sigma^2} \]

\[ = \frac{\frac{t^n}{2^{n/2} \left( \frac{1}{\sigma^2} \right)^{n/2}}}{\Gamma \left( \frac{n}{2} \right) 2^{n/2} \left( \frac{1}{\sigma^2} \right)^{n/2}} e^{-\frac{t}{2\sigma^2}} \]
Normal Example with Known Mean (cont’d)

\[
f_T(t) = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{t}{\sigma^2} \right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}} \left| \frac{dy}{dt} \right|
\]

\[
= \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{t}{\sigma^2} \right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}} \frac{1}{\sigma^2}
\]

\[
= \frac{t^{\frac{n}{2}-1}}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{1}{\sigma^2} \right)^{\frac{n}{2}} e^{-\frac{t}{2\sigma^2}}
\]

\[
= h(t) c(\sigma^2) \exp[w(\sigma^2) t]
\]

where \( w(\sigma^2) = -\frac{1}{2\sigma^2} \) is an increasing function in \( \sigma^2 \).
Normal Example with Known Mean (cont’d)

\[ f_T(t) = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{t}{\sigma^2} \right)^{n/2-1} e^{-\frac{t}{2\sigma^2}} \left| \frac{dy}{dt} \right| \]

\[ = \frac{1}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{t}{\sigma^2} \right)^{n/2-1} e^{-\frac{t}{2\sigma^2}} \frac{1}{\sigma^2} \]

\[ = \frac{t^{n/2-1}}{\Gamma \left( \frac{n}{2} \right) 2^{n/2}} \left( \frac{1}{\sigma^2} \right)^{n/2} e^{-\frac{t}{2\sigma^2}} \]

\[ = h(t) c(\sigma^2) \exp[w(\sigma^2)t] \]

where \( w(\sigma^2) = -\frac{1}{2\sigma^2} \) is an increasing function in \( \sigma^2 \). Therefore, \( T = \sum_{i=1}^{n} (X_i - \mu_0)^2 \) has the MLR property.
Normal Example with Known Mean (cont’d)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s H_0$ if and only if $T > t_0$ where $t_0$ is chosen such that $\alpha = \Pr(T > t_0 | \sigma_0^2)$. 
Normal Example with Known Mean (cont’d)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s \; H_0$ if and only if $T > t_0$ where $t_0$ is chosen such that $\alpha = \Pr(T > t_0 | \sigma_0^2)$. Note that $\frac{T}{\sigma^2} \sim \chi^2_n$

$$\Pr(T > t_0 | \sigma_0^2) = \Pr \left( \frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2} \Bigg| \sigma_0^2 \right)$$
Normal Example with Known Mean (cont’d)

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Note that $\frac{T}{\sigma^2} \sim \chi^2_n$

$$\Pr(T > t_0 | \sigma^2_0) = \Pr \left( \frac{T}{\sigma^2_0} > \frac{t_0}{\sigma^2_0} \bigg| \sigma^2_0 \right)$$

$$\frac{T}{\sigma^2} \sim \chi^2_n$$
Normal Example with Known Mean (cont’d)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s H_0$ if and only if $T > t_0$ where $t_0$ is chosen such that $\alpha = \Pr(T > t_0|\sigma_0^2)$.

Note that $\frac{T}{\sigma^2} \sim \chi_n^2$

$$\Pr(T > t_0|\sigma_0^2) = \Pr\left(\frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2} \mid \sigma_0^2\right)$$

$$\frac{T}{\sigma_0^2} \sim \chi_n^2$$

$$\Pr\left(\chi_n^2 > \frac{t_0}{\sigma_0^2}\right) = \alpha$$
Normal Example with Known Mean (cont’d)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s H_0$ if and only if $T > t_0$ where $t_0$ is chosen such that $\alpha = Pr(T > t_0|\sigma^2_0)$.

Note that $\frac{T}{\sigma^2} \sim \chi^2_n$

$$Pr(T > t_0|\sigma^2_0) = Pr\left(\frac{T}{\sigma^2_0} > \frac{t_0}{\sigma^2_0}\right|\sigma^2_0)$$

$$\frac{T}{\sigma^2_0} \sim \chi^2_n$$

$$Pr\left(\chi^2_n > \frac{t_0}{\sigma^2_0}\right) = \alpha$$

$$\frac{t_0}{\sigma^2_0} = \chi^2_{n,\alpha}$$
Normal Example with Known Mean (cont’d)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s \, H_0$ if and only if $T > t_0$
where $t_0$ is chosen such that $\alpha = \Pr(T > t_0|\sigma_0^2)$.

Note that $\frac{T}{\sigma^2} \sim \chi^2_n$

\[
\Pr(T > t_0|\sigma_0^2) = \Pr \left( \frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2} \mid \sigma_0^2 \right)
\]

\[
\frac{T}{\sigma_0^2} \sim \chi^2_n
\]

\[
\Pr \left( \chi^2_n > \frac{t_0}{\sigma_0^2} \right) = \alpha
\]

\[
\frac{t_0}{\sigma_0^2} = \chi^2_{n, \alpha}
\]

\[
t_0 = \sigma_0^2 \chi^2_{n, \alpha}
\]

where $\chi^2_{n, \alpha}$ satisfies $\int_{\chi^2_{n, \alpha}}^{\infty} f_{\chi^2_n}(x) \, dx = \alpha$. 
Remarks

- For many problems, UMP level $\alpha$ test does not exist (Example 8.3.19).
For many problems, UMP level $\alpha$ test does not exist (Example 8.3.19).

In such cases, we can restrict our search among a subset of tests, for example, all unbiased tests.
Summary

Today

- Uniformly Most Powerful Test
- Neyman-Pearson Lemma
- Monotone Likelihood Ratio
- Karlin-Rabin Theorem
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- Uniformly Most Powerful Test
- Neyman-Pearson Lemma
- Monotone Likelihood Ratio
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Next Lecture

- Asymptotics of LRT
- Wald Test