Biostatistics 602 - Statistical Inference
Lecture 11
Evaluation of Point Estimators

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February 14th, 2013

Some News

- Homework 3 is posted.
  - Due is Tuesday, February 26th.

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    - Due is Tuesday, February 26th.
- Next Thursday (Feb 21) is the midterm day.
    - We will start sharply at 1:10pm.
    - It would be better to solve homework 3 yourself to get prepared.
    - The exam is closed book, covering all the material from Lecture 1 to 12.
    - Last year's midterm is posted on the web page.

Last Lecture

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1. What is a maximum likelihood estimator (MLE)?
2. How can you find an MLE?
3. Does an ML estimate always fall into a valid parameter space?
4. If you know MLE of θ, can you also know MLE of $\tau(\theta)$?

Recap - Maximum Likelihood Estimator

Definition

- For a given sample point $\mathbf{x} = (x_1, \cdots, x_n)$,
- let $\hat{\theta}(\mathbf{x})$ be the value such that
- $L(\theta|\mathbf{x})$ attains its maximum.
- More formally, $L(\hat{\theta}(\mathbf{x})|\mathbf{x}) \geq L(\theta|\mathbf{x})\ \forall\theta \in \Omega$ where $\hat{\theta}(\mathbf{x}) \in \Omega$.
- $\hat{\theta}(\mathbf{x})$ is called the *maximum likelihood estimate* of θ based on data $\mathbf{x}$,
- and $\hat{\theta}(\mathbf{X})$ is the *maximum likelihood estimator (MLE)* of θ.

Recap - Invariance Property of MLE

Question

If $\hat{\theta}$ is the MLE of θ, what is the MLE of $\tau(\theta)$?

Example

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ where $0 < p < 1$.

1. What is the MLE of p?
2. What is the MLE of odds, defined by $\eta = p/(1-p)$?

Getting MLE of $\eta = \frac{p}{1-p}$ from $\hat{p}$

$$L^*(\eta|\mathbf{x}) = \frac{\eta^{\sum x_i}}{(1+\eta)^n}$$

- From MLE of $\hat{p}$, we know $L^*(\eta|\mathbf{x})$ is maximized when $p = \eta/(1+\eta) = \hat{p}$.
- Equivalently, $L^*(\eta|\mathbf{x})$ is maximized when $\eta = \hat{p}/(1-\hat{p}) = \tau(\hat{p})$, because τ is a one-to-one function.
- Therefore $\hat{\eta} = \tau(\hat{p})$.

Invariance Property of MLE

Fact

Denote the MLE of θ by $\hat{\theta}$. If $\tau(\theta)$ is an one-to-one function of θ, then MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$.

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The likelihood function in terms of $\tau(\theta) = \eta$ is

$$L^*(\tau(\theta)|\mathbf{x}) = \prod_{i=1}^{n} f_X(x_i|\theta)$$

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$$\begin{aligned} L^*(\tau(\theta)|\mathbf{x}) &= \prod_{i=1}^{n} f_X(x_i|\theta) = \prod_{i=1}^{n} f(x_i|\tau^{-1}(\eta)) \\ &= L(\tau^{-1}(\eta)|\mathbf{x}) \end{aligned}$$

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We know this function is maximized when $\tau^{-1}(\eta) = \hat{\theta}$, or equivalently, when $\eta = \tau(\hat{\theta})$. Therefore, MLE of $\eta = \tau(\theta)$ is $\tau(\hat{\theta})$.

Induced Likelihood Function

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We define the *induced likelihood function* L^* by

$$L^*(\eta|\mathbf{x}) = \sup_{\theta \in \tau^{-1}(\eta)} L(\theta|\mathbf{x})$$

where $\tau^{-1}(\eta) = \{\theta : \tau(\theta) = \eta,\ \theta \in \Omega\}$.

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where $\tau^{-1}(\eta) = \{\theta : \tau(\theta) = \eta, \ \theta \in \Omega\}$.

- The value of η that maximize $L^*(\eta|\mathbf{x})$ is called the MLE of $\eta = \tau(\theta)$.

Invariance Property of MLE

Theorem 7.2.10

If θ is the MLE of $\hat{\theta}$, then the MLE of $\eta = \tau(\theta)$ is $\tau(\hat{\theta})$, where $\tau(\theta)$ is any function of θ.

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Proof - Using Induced Likelihood Function

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Hence, $L^*(\hat{\eta}|\mathbf{x}) = L^*[\tau(\hat{\theta})|\mathbf{x}]$ and $\tau(\hat{\theta})$ is the MLE of $\tau(\theta)$.

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2. By definition, MLE will always fall into the range of the parameter space.
3. Not always easy to obtain; may be hard to find the global maximum.
4. Heavily depends on the underlying distributional assumptions (i.e. not robust).

Method of Evaluating Estimators

Definition : Unbiasedness

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Example

$X_1, \cdots, X_n$ are iid samples from a distribution with mean μ. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an estimator of μ.

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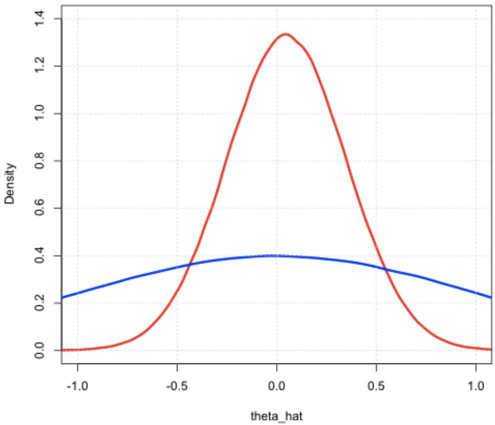
Therefore $\overline{X}$ is an unbiased estimator for μ.

How important is unbiased?



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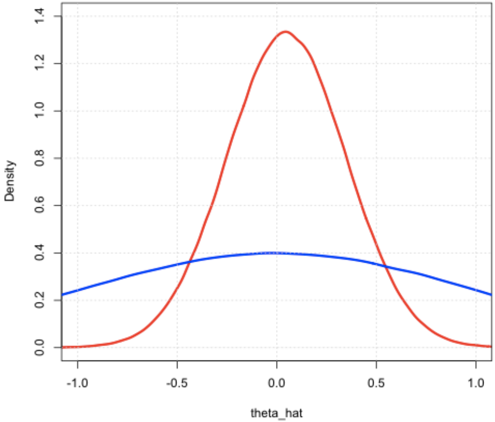

- $\hat{\theta}_1$ (blue) is unbiased but has a chance to be very far away from $\theta = 0$.
- $\hat{\theta}_2$ (red) is biased but more likely to be closer to the true θ than $\hat{\theta}_1$.

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&= \text{Var}(\hat{\theta}) + \text{Bias}^2(\theta)
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- Therefore, we cannot find an estimator that is uniformly the best in terms of MSE across all $\theta \in \Omega$ among all estimators
- Restrict the class of estimators, and find the "best" estimator within the small class.

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How to find the Best Unbiased Estimator

- Find the lower bound of variances of any unbiased estimator of $\tau(\theta)$, say $B(\theta)$.
- If W^* is an unbiased estimator of $\tau(\theta)$ and satisfies $\text{Var}[W^*(\mathbf{X})|\theta] = B(\theta)$, then W^* is the best unbiased estimator.

Cramer-Rao inequality

Theorem 7.3.9 : Cramer-Rao Theorem

Let $X_1, \cdots, X_n$ be a sample with joint pdf/pmf of $f_{\mathbf{X}}(\mathbf{x}|\theta)$. Suppose $W(\mathbf{X})$ is an estimator satisfying

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1. $E[W(\mathbf{X})|\theta] = \tau(\theta), \ \forall\theta \in \Omega.$

Cramer-Rao inequality

Theorem 7.3.9 : Cramer-Rao Theorem

Let $X_1, \cdots, X_n$ be a sample with joint pdf/pmf of $f_{\mathbf{X}}(\mathbf{x}|\theta)$. Suppose $W(\mathbf{X})$ is an estimator satisfying

1. $E[W(\mathbf{X})|\theta] = \tau(\theta), \ \forall\theta \in \Omega$.
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Then, a lower bound of $\text{Var}[W(\mathbf{X})|\theta]$ is

$$\text{Var}[W(\mathbf{X})] \geq \frac{[\tau'(\theta)]^2}{E\left[\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)\}^2\right]}$$

Proving Cramer-Rao Theorem (1/4)

By Cauchy-Schwarz inequality,

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Using $\mathrm{Var}(X) = EX^2 - (EX)^2$,

$$\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] - E\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right]^2$$

Proving Cramer-Rao Theorem (2/4)

$$E\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = \int_{\mathbf{x}\in\mathcal{X}}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)\right] f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x}$$

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Proving Cramer-Rao Theorem (2/4)

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\begin{aligned}
E\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= \int_{\mathbf{x}\in\mathcal{X}}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)\right] f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \int_{\mathbf{x}\in\mathcal{X}}\frac{\frac{\partial}{\partial\theta}f_{\mathbf{X}}(\mathbf{x}|\theta)}{f_{\mathbf{X}}(\mathbf{x}|\theta)} f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \int_{\mathbf{x}\in\mathcal{X}}\frac{\partial}{\partial\theta}f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x}
\end{aligned}
$$

Proving Cramer-Rao Theorem (2/4)

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&= \int_{\mathbf{x}\in\mathcal{X}} \frac{\partial}{\partial\theta} f_{\mathbf{X}}(\mathbf{x}|\theta)\, d\mathbf{x} \\
&= \frac{d}{d\theta}\int_{\mathbf{x}\in\mathcal{X}} f_{\mathbf{X}}(\mathbf{x}|\theta)\, d\mathbf{x} \qquad \text{(by assumption)}
\end{aligned}
$$

Proving Cramer-Rao Theorem (2/4)

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E\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= \int_{\mathbf{x}\in\mathcal{X}}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)\right] f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
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&= \int_{\mathbf{x}\in\mathcal{X}}\frac{\partial}{\partial\theta}f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \frac{d}{d\theta}\int_{\mathbf{x}\in\mathcal{X}} f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \qquad \text{(by assumption)} \\
&= \frac{d}{d\theta}1 = 0
\end{aligned}$$

Proving Cramer-Rao Theorem (2/4)

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E\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= \int_{\mathbf{x}\in\mathcal{X}}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)\right] f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
&= \int_{\mathbf{x}\in\mathcal{X}} \frac{\frac{\partial}{\partial\theta} f_{\mathbf{X}}(\mathbf{x}|\theta)}{f_{\mathbf{X}}(\mathbf{x}|\theta)} f_{\mathbf{X}}(\mathbf{x}|\theta)\,d\mathbf{x} \\
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&= \frac{d}{d\theta} 1 = 0 \\
\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right]
\end{aligned}
$$

Proving Cramer-Rao Theorem (3/4)

$$\text{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right]$$

Proving Cramer-Rao Theorem (3/4)

$$
\begin{aligned}
& \mathrm{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] \\
= \;& E\left[W(\mathbf{X}) \cdot \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] - E\left[W(\mathbf{X}) \right] E\left[\frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right]
\end{aligned}
$$

Proving Cramer-Rao Theorem (3/4)

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= \; & E\left[W(\mathbf{X}) \cdot \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right]
\end{aligned}
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= \; & E\left[W(\mathbf{X}) \cdot \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = \int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x})\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{x}|\theta)f(\mathbf{x}|\theta)\,d\mathbf{x}
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&\text{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] \\
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=\;& \int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x})\frac{\frac{\partial}{\partial\theta}f_{\mathbf{X}}(\mathbf{x}|\theta)}{f(\mathbf{x}|\theta)}f(\mathbf{x}|\theta)\,d\mathbf{x} = \int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x})\frac{\partial}{\partial\theta}f_{\mathbf{X}}(\mathbf{x}|\theta)
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& \text{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] \\
= \; & E\left[W(\mathbf{X}) \cdot \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] - E\left[W(\mathbf{X}) \right] E\left[\frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] \\
= \; & E\left[W(\mathbf{X}) \cdot \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta) \right] = \int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x}) \frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{x}|\theta) f(\mathbf{x}|\theta)\, d\mathbf{x} \\
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= \; & \frac{d}{d\theta} \int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}|\theta) \qquad \text{(by assumption)}
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Proving Cramer-Rao Theorem (3/4)

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= \; & \frac{d}{d\theta}\int_{\mathbf{x}\in\mathcal{X}} W(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}|\theta) \qquad \text{(by assumption)} \\
= \; & \frac{d}{d\theta} E\left[W(\mathbf{X})\right] = \frac{d}{d\theta}\tau(\theta) = \tau'(\theta)
\end{aligned}
$$

Proving Cramer-Rao Theorem (4/4)

From the previous results

$$\text{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right]$$

Proving Cramer-Rao Theorem (4/4)

From the previous results

$$\begin{aligned} \mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] \\ \mathrm{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] &= \tau'(\theta) \end{aligned}$$

Therefore, Cramer-Rao lower bound is

$$\mathrm{Var}[W(\mathbf{X})] \geq \frac{\left[\mathrm{Cov}\{W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\}\right]^2}{\mathrm{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right]}$$

Proving Cramer-Rao Theorem (4/4)

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$$\text{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right]$$

$$\text{Cov}\left[W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right] = \tau'(\theta)$$

Therefore, Cramer-Rao lower bound is

$$\begin{aligned}\text{Var}[W(\mathbf{X})] &\geq \frac{\left[\text{Cov}\{W(\mathbf{X}), \frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\}\right]^2}{\text{Var}\left[\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right]} \\ &= \frac{[\tau'(\theta)]^2}{E\left[\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\}^2\right]}\end{aligned}$$

Cramer-Rao bound in iid case

Corollary 7.3.10

If $X_1, \cdots, X_n$ are iid samples from pdf/pmf $f_X(x|\theta)$, and the assumptions in the above Cramer-Rao theorem hold, then the lower-bound of $\mathrm{Var}[W(\mathbf{X})|\theta]$ becomes

Cramer-Rao bound in iid case

Corollary 7.3.10

If $X_1, \cdots, X_n$ are iid samples from pdf/pmf $f_X(x|\theta)$, and the assumptions in the above Cramer-Rao theorem hold, then the lower-bound of $\text{Var}[W(\mathbf{X})|\theta]$ becomes

$$\text{Var}[W(\mathbf{X})] \geq \frac{[\tau'(\theta)]^2}{nE\left[\{\frac{\partial}{\partial\theta}\log f_X(X|\theta)\}^2\right]}$$

Cramer-Rao bound in iid case

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If $X_1, \cdots, X_n$ are iid samples from pdf/pmf $f_X(x|\theta)$, and the assumptions in the above Cramer-Rao theorem hold, then the lower-bound of $\mathrm{Var}[W(\mathbf{X})|\theta]$ becomes

$$\mathrm{Var}[W(\mathbf{X})] \geq \frac{[\tau'(\theta)]^2}{nE\left[\{\frac{\partial}{\partial\theta}\log f_X(X|\theta)\}^2\right]}$$

Proof

We need to show that

$$E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] = nE\left[\left\{\frac{\partial}{\partial\theta}\log f_X(X|\theta)\right\}^2\right]$$

Proving Corollary 7.3.10

$$E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] = E\left[\left\{\frac{\partial}{\partial\theta}\log\prod_{i=1}^{n} f_X(X_i|\theta)\right\}^2\right]$$

Proving Corollary 7.3.10

$$E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] = E\left[\left\{\frac{\partial}{\partial\theta}\log\prod_{i=1}^n f_X(X_i|\theta)\right\}^2\right]$$

$$= E\left[\left\{\frac{\partial}{\partial\theta}\sum_{i=1}^n \log f_X(X_i|\theta)\right\}^2\right]$$

Proving Corollary 7.3.10

$$
\begin{aligned}
E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] &= E\left[\left\{\frac{\partial}{\partial\theta}\log\prod_{i=1}^n f_X(X_i|\theta)\right\}^2\right] \\
&= E\left[\left\{\frac{\partial}{\partial\theta}\sum_{i=1}^n \log f_X(X_i|\theta)\right\}^2\right] \\
&= E\left[\left\{\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f_X(X_i|\theta)\right\}^2\right]
\end{aligned}
$$

Proving Corollary 7.3.10

$$
\begin{aligned}
E\left[\left\{\frac{\partial}{\partial\theta}\log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] &= E\left[\left\{\frac{\partial}{\partial\theta}\log\prod_{i=1}^n f_X(X_i|\theta)\right\}^2\right] \\
&= E\left[\left\{\frac{\partial}{\partial\theta}\sum_{i=1}^n \log f_X(X_i|\theta)\right\}^2\right] \\
&= E\left[\left\{\sum_{i=1}^n \frac{\partial}{\partial\theta}\log f_X(X_i|\theta)\right\}^2\right] \\
&= E\Big[\textstyle\sum_{i=1}^n \left\{\frac{\partial}{\partial\theta}\log f_X(X_i|\theta)\right\}^2 + \\
&\quad \textstyle\sum_{i\neq j}\frac{\partial}{\partial\theta}\log f_X(X_i|\theta)\frac{\partial}{\partial\theta}\log f_X(X_j|\theta)\Big]
\end{aligned}
$$

Proving Corollary 7.3.10

Because $X_1, \cdots, X_n$ are independent,

$$E\left[\sum_{i \neq j} \frac{\partial}{\partial\theta} \log f_X(X_i|\theta) \frac{\partial}{\partial\theta} \log f_X(X_j|\theta)\right]$$

$$= \sum_{i \neq j} E\left[\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right] E\left[\frac{\partial}{\partial\theta} \log f_X(X_j|\theta)\right] = 0$$

Proving Corollary 7.3.10

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$$E\left[\left\{\frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] = E\left[\sum_{i=1}^{n} \left\{\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right\}^2\right]$$

Proving Corollary 7.3.10

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$$\begin{aligned} E\left[\left\{\frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] &= E\left[\sum_{i=1}^n \left\{\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right\}^2\right] \\ &= \sum_{i=1}^n E\left[\left\{\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right\}^2\right] \end{aligned}$$

Proving Corollary 7.3.10

Because $X_1, \cdots, X_n$ are independent,

$$E\left[\sum_{i \neq j} \frac{\partial}{\partial\theta} \log f_X(X_i|\theta) \frac{\partial}{\partial\theta} \log f_X(X_j|\theta)\right]$$

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$$\begin{aligned} E\left[\left\{\frac{\partial}{\partial\theta} \log f_{\mathbf{X}}(\mathbf{X}|\theta)\right\}^2\right] &= E\left[\sum_{i=1}^n \left\{\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right\}^2\right] \\ &= \sum_{i=1}^n E\left[\left\{\frac{\partial}{\partial\theta} \log f_X(X_i|\theta)\right\}^2\right] \\ &= nE\left[\left\{\frac{\partial}{\partial\theta} \log f_X(X|\theta)\right\}^2\right] \end{aligned}$$

Remark from Corollary 7.3.10

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Because $\tau(\theta) = \theta$ and $\tau'(\theta) = 1$.

Score Function

Definition: Score or Score Function for X

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The bigger the information number, the more information we have about θ, the smaller bound on the variance of unbiased estimates.

Simplified Fisher Information

Lemma 7.3.11

If $f_X(x|\theta)$ satisfies the two interchangeability conditions

$$\frac{d}{d\theta}\int_{x\in\mathcal{X}} f_X(x|\theta)\,dx = \int_{x\in\mathcal{X}} \frac{\partial}{\partial\theta} f_X(x|\theta)\,dx$$

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which are true for exponential family, then

$$I(\theta) = E\left[\left\{\frac{\partial}{\partial\theta}\log f_X(X|\theta)\right\}^2\right] = -E\left[\frac{\partial^2}{\partial\theta^2}\log f_X(X|\theta)\right]$$

Example - Poisson Distribution

- $X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda)$
- $\lambda_1 = \overline{X}$
- $\lambda_2 = s_{\mathbf{X}}^2$
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Example - Poisson Distribution (cont'd)

Therefore, the Cramer-Rao lower bound is

$$\mathrm{Var}[W(\mathbf{X})] \geq \frac{1}{nI(\lambda)} = \frac{\lambda}{n}$$

where W is any unbiased estimator.

Example - Poisson Distribution (cont'd)

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Therefore, $\lambda_1 = \overline{X}$ is the best unbiased estimator of λ.

$$\mathrm{Var}(\hat{\lambda}_2) > \frac{\lambda}{n}$$

(details is omitted), so $\hat{\lambda}_2$ is not the best unbiased estimator.

With and without Lemma 7.3.11

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$$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\log f_X(X|\lambda)\right] = -E\left[\frac{\partial^2}{\partial\lambda^2}\left(-\lambda + X\log\lambda - \log X!\right)\right] = \frac{1}{\lambda}$$

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&= 1 - 2\frac{E(X)}{\lambda} + \frac{\mathrm{Var}(X) + [E(X)]^2}{\lambda^2} = 1 - 2\frac{\lambda}{\lambda} + \frac{\lambda+\lambda^2}{\lambda^2} = \frac{1}{\lambda}
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Applying Lemma 7.3.11

Question

When can we interchange the order of differentiation and integration?

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$$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$$

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Answer

- For exponential family, always yes.
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Example

$X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$

$$\frac{d}{d\theta} \int_0^\theta h(x) f_X(x|\theta)\, dx \neq \int_0^\theta h(x) \frac{\partial}{\partial\theta} f_X(x|\theta)\, dx$$

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Today

- Invariance Property
- Mean Squared Error
- Unbiased Estimator
- Cramer-Rao inequality

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Next Lecture

- More on Cramer-Rao inequality