Biostatistics 602 - Statistical Inference
Lecture 09
Likelihood and Point Estimation

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Likelihood Function

Definition

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} f_X(x|\theta)$. The join distribution of $\mathbf{X} = (X_1, \cdots, X_n)$ is

$$f_{\mathbf{X}}(\mathbf{x}|\theta) = \prod_{i=1}^{n} f_X(x_i|\theta)$$

Given that $\mathbf{X} = \mathbf{x}$ is observed, the function of θ defined by $L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$ is called the likelihood function.

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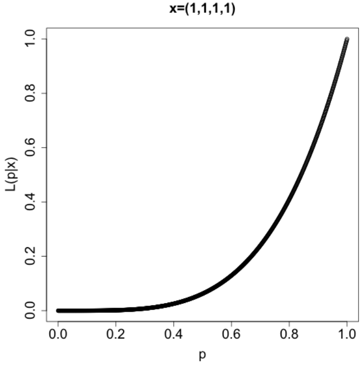

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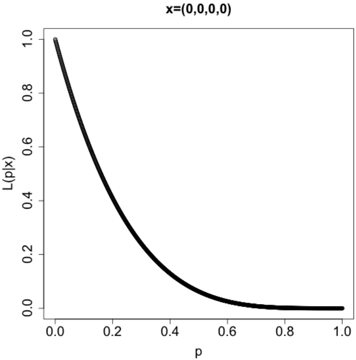

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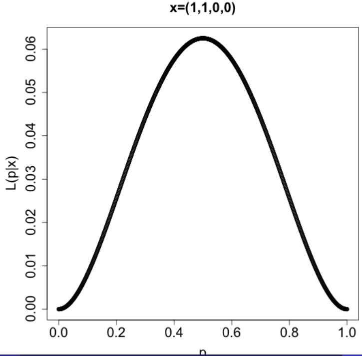

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- Task is to use data $\mathbf{x}$ to make inference on θ

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- Define $w_2(X_1, \cdots, X_n) = X_{(1)} = 1.2$.

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Solving the two equations above, $\hat{\mu} = \overline{X}$, $\hat{\sigma^2} = \sum_{i=1}^n (X_i - \overline{X})^2/n$.

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$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mu_2' = E[\mathbf{X}^2] = (E\mathbf{X})^2 + \text{Var}(\mathbf{X}) = k^2p^2 + kp(1-p)$$

Method of moments estimator - Binomial (cont'd)

The method of moments estimators are

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These are not the best estimators. It is possible to get negative estimates of k and p.

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\hat{p} &= \frac{m_1}{m_2 - m_1^2} = \frac{\overline{X}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2}
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\hat{r} &= \frac{m_1 \hat{p}}{1-\hat{p}} = \frac{\overline{X}\hat{p}}{1-\hat{p}}
\end{aligned}$$

Satterthwaite Approximation

Problem

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However, the distribution of $\sum_{i=1}^k a_i Y_i$, where a_is are known constants with $\sum_{i=1}^n a_i r_i = 1$, in general, the distribution is hard to obtain.

It is often reasonable to assume that the distribution of $\sum_{i=1}^k a_i Y_i$ follows $\frac{1}{\nu}\chi^2_\nu$ approximately. Find a moment-based estimator of ν.

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Therefore, the method of moment estimator of ν is

$$\hat{\nu} = \frac{2}{\left(\sum_{i=1}^k a_i Y_i\right)^2 - 1}$$

Note that ν can be negative, which is not desirable.

An alternative Solution

To match the second moments,

$$E\left(\sum_{i=1}^{k} a_i Y_i\right)^2 = \mathrm{Var}\left(\sum_{i=1}^{k} a_i Y_i\right) + \left[E(\sum_{i=1}^{k} a_i Y_i)\right]^2$$

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E\left(\sum_{i=1}^{k} a_i Y_i\right)^2 &= \text{Var}\left(\sum_{i=1}^{k} a_i Y_i\right) + \left[E(\sum_{i=1}^{k} a_i Y_i)\right]^2 \\
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An alternative Solution

To match the second moments,

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Substituting this expression for the variance and removing expectations, we obtain Satterthwaite's estimator

$$\hat{\nu} = \frac{\sum_{i=1}^{n} a_i Y_i}{\sum_{i=1}^{n} \frac{a_i^2}{r_i} Y_i^2}$$

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- and $\hat{\theta}(\mathbf{X})$ is the *maximum likelihood estimator (MLE)* of θ.

Example of MLE - Exponential Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\beta)$. Find MLE of β.

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\begin{aligned}
L(\beta|\mathbf{x}) &= f_{\mathbf{X}}(\mathbf{x}|\theta) = \prod_{i=1}^{n} f_X(x_i|\theta) \\
&= \prod_{i=1}^{n} \left[\frac{1}{\beta} e^{-x_i/\beta} \right] = \frac{1}{\beta^n} \exp\left(-\sum_{i=1}^{n} \frac{x_i}{\beta} \right)
\end{aligned}
$$

where $\beta > 0$.

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\sum_{i=1}^n x_i &= n\beta \\
\hat{\beta} &= \frac{\sum_{i=1}^n x_i}{n} = \bar{x}
\end{aligned}
$$

Use the double derivative to confirm local maximum

$$\left.\frac{\partial^2 l}{\partial \beta^2}\right|_{\beta=\bar{x}} = \left. -2\frac{\sum_{i=1}^n x_i}{\beta^3} + \frac{n}{\beta^2}\right|_{\beta=\bar{x}}$$

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Therefore, we can conclude that $\hat{\beta}(\mathbf{X}) = \overline{X}$ is unique local maximum on the interval

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Therefore $l(\beta|\mathbf{x})$ and $L(\beta|\mathbf{x})$ attains the global maximum when $\hat{\beta} = \overline{x}$
$\hat{\beta}(\mathbf{X}) = \overline{X}$ is the MLE of β.

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   - Check boundary points to see whether boundary gives global maximum.

If the function is NOT differentiable with respect to θ.

- Use numerical methods
- Or perform directly maximization, using inequalities, or properties of the function.

Summary

Today

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- Point Estimator
- Method of Moments Estimator
- Maximum Likelihood Estimator

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Next Lecture

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