# Biostatistics 602 - Statistical Inference
# Lecture 14
# Obtaining Best Unbiased Estimator

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# Last Lecture

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- What is the Rao-Blackwell Theorem?
- Is the best unbiased estimator (UMVUE) for $\tau(\theta)$ is unique?
- What is the relationship between the UMVUE and the unbiased estimators of zero?

# Rao-Blackwell Theorem

## Theorem 7.3.17

Let $W(\mathbf{X})$ be any unbiased estimator of $\tau(\theta)$, and $T$ be a sufficient statistic for $\theta$. Define $\phi(T) = E[W|T]$. Then the followings hold.

1. $E[\phi(T)|\theta] = \tau(\theta)$
2. $\mathrm{Var}[\phi(T)|\theta] \leq \mathrm{Var}(W|\theta)$ for all $\theta$.

That is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

# Related Theorems

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**Theorem 7.3.20 - UMVUE and unbiased estimators of zero**

If $E[W(\mathbf{X})] = \tau(\theta)$. $W$ is the best unbiased estimator of $\tau(\theta)$ if an only if $W$ is uncorrelated with all unbiased estimator of $0$.

# The power of complete sufficient statistics

**Theorem 7.3.23**

Let $T$ be a complete sufficient statistic for parameter $\theta$. Let $\phi(T)$ be any estimator based on $T$. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.

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- $W(\mathbf{X})$ : unbiased for $\tau(\theta)$.
- $T^*(\mathbf{X})$ : sufficient statistic for $\theta$.

$\phi(T) = E[W(\mathbf{X})|T(\mathbf{X})]$ is a better unbiased estimator of $\tau(\theta)$.

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In fact, we only need to consider functions of minimal sufficient statistics to find the best unbiased estimator.

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Let $T(\mathbf{X})$ be a minimal sufficient, and $T^*(\mathbf{X})$ be a sufficient statistic. Then by definition, there exists a function $h$ that satisfies $T = h(T^*)$.

$$\begin{aligned} E[\phi(T)|T^*] &= E[\phi\{h(T^*)\}|T^*] \\ &= \phi\{h(T^*)\} = \phi(T) \end{aligned}$$

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$$\begin{aligned} E[\phi(T)|T^*] &= E[\phi\{h(T^*)\}|T^*] \\ &= \phi\{h(T^*)\} = \phi(T) \end{aligned}$$

Therefore $\phi(T)$ remains the same after conditioning on any sufficient statistic $T^*$.

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- Suppose that $T$ is a complete statistic, then $U(T)$ can only be zero almost surely.

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- By definition, $T$ is complete is $E[U(T)] = 0$ for all $\theta$ implies $U(T) = 0$ almost surely.
- Suppose that $T$ is a complete statistic, then $U(T)$ can only be zero almost surely.
- Therefore, $\mathrm{Cov}(\phi(T), U(T)) = \mathrm{Cov}(\phi(T), 0) = 0$, and $\phi(T)$ is the best unbiased estimator of its expected value (Theorem 7.3.23).

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   - Guess a functon $\phi(T)$ such that $E[\phi(T)] = \tau(\theta)$.
   - Guess an unbiased estimator $h(\mathbf{X})$ of $\tau(\theta)$. Construct $\phi(T) = E[h(\mathbf{X})|T]$, then $E[\phi(T)] = E[h(\mathbf{X})] = \tau(\theta)$.

# Example - Normal Distribution

## Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$. Find the best unbiased estimator for (1) $\mu$, (2) $\sigma^2$, (3) $\mu^2$.

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- Because $E[\overline{X}] = \mu$, $\overline{X}$ is an unbiased estimator for $\mu$, $\overline{X}$ is also a function of $\mathbf{T}(\mathbf{X})$.
- Therefore, $\overline{X}$ is the best unbiased estimator for $\mu$.

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To obtain UMVUE for $\mu^2$, we need a $\phi(\mathbf{T}) = \phi(\overline{X}, s^2_{\mathbf{X}})$ such that $E[\phi(T)] = \mu^2$.

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- $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is unbiased estimator for $\mu^2$
- And it is a function of $(\overline{X}, s^2_{\mathbf{X}})$.
- Hence, $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is the best unbiased estimator for $\mu^2$.

# Example - Normal Distribution - Alternative method

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&= \frac{E[\left(n\overline{X}\right)^2 - (n-1)s_{\mathbf{X}}^2 - n\overline{X}^2|\mathbf{T}]}{n(n-1)}
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&= \overline{X}^2 - s_{\mathbf{X}}^2 / n
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# Example - Uniform Distribution

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$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$. Find the best unbiased estimator for (1) $\theta$, (2) $g(\theta)$ differentiable on $(0, \theta)$ (3) $\theta^2$, (4) $1/\theta$.

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## Solution - MVUE of $\theta$

- $T(\mathbf{X}) = X_{(n)}$ is a complete and sufficient statistic for $\theta$.

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- $E[\phi(T)] = E\left[\frac{n+1}{n} X_{(n)}\right] = \theta$.

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- $E[\phi(T)] = E\left[\frac{n+1}{n} X_{(n)}\right] = \theta$.

$\frac{n+1}{n} X_{(n)}$ is the best unbiased estimator of $\theta$.

# Example - Uniform Distribution - for $g(\theta)$

We need to find a function of $\phi(T) = X_{(n)}$ such that $E[\phi(T)] = g(\theta)$.

$$g(\theta) = E[\phi(T)] = \int_0^\theta \phi(t) n \theta^{-n} t^{n-1} dt$$

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Taking derivative with respect to $\theta$, and applying Leibnitz's rule.

$$g'(\theta) = \frac{d}{d\theta} \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt$$

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# Example - Uniform Distribution - for $g(\theta)$ (cont'd)

$$g'(\theta) = \phi(\theta) n\theta^{-1} - n\theta^{-1} \int_0^\theta \phi(t) n t^{n-1} \theta^{-n} dt$$

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Therefore, the best unbiased estimator of $g(\theta)$ is

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# Example - Binomial best unbiased estimator

## Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binomial}(k, \theta)$. Estimate the probability of exactly one success.

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- So we need to find a $\phi(T)$ that satisfies $E[\phi(T)] = \tau(\theta)$.
- There is no imeediately evident unbiased estimator of $\tau(\theta)$ as a function of $T$.

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- Start with a simple-minded estimator

$$W(X_1) = \begin{cases} 1 & X_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

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and hence is an unbiased estimator of $\tau(\theta) = k\theta(1-\theta)^{k-1}$.

- The best unbiased estimator of $\tau(\theta)$ is

$$\phi(T) = E[W|T] = E[W(X_1)|T(\mathbf{X})]$$

# Solution - Binomial best unbiased estimator (cont'd)

$$\phi(t) = E\left[W(X_1)|\sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1|\sum_{i=1}^{n} X_i = t\right]$$

# Solution - Binomial best unbiased estimator (cont'd)

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&= \frac{[k\theta(1-\theta)^{k-1}]\left[\binom{k(n-1)}{t-1}\theta^{t-1}(1-\theta)^{k(n-1)-t-1}\right]}{\binom{kn}{n}\theta^{t}(1-\theta)^{kn-t}}
\end{aligned}
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\end{aligned}
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# Solution - Binomial best unbiased estimator (cont'd

Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

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Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

$$\phi\left(\sum_{i=1}^{n} X_i\right) = k\frac{\binom{k(n-1)}{\sum X_i - 1}}{\binom{kn}{\sum X_i}}$$

# Summary

## Today

- Rao-Blackwell Theorem
- Methods for obtaining UMVUE

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## Next Lecture

- Bayesian Estimators