Biostatistics 602 - Statistical Inference
Lecture 22
p-Values

Hyun Min Kang

April 9th, 2013
Last Lecture

• Is the exact distribution of LRT statistic typically easy to obtain?
• How about its asymptotic distribution? For testing which null/alternative hypotheses is the asymptotic distribution valid?
• What is a Wald Test?
• Describe a typical way to construct a Wald Test.
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Describe a typical way to construct a Wald Test.
Asymptotics of LRT

Theorem 10.3.1

Consider testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose $X_1, \cdots, X_n$ are iid samples from $f(x|\theta)$, and $\hat{\theta}$ is the MLE of $\theta$, and $f(x|\theta)$ satisfies certain "regularity conditions" (e.g. see misc 10.6.2), then under $H_0$: 
Asymptotics of LRT

Theorem 10.3.1
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$$-2 \log \lambda(x) \xrightarrow{d} \chi_1^2$$

as $n \to \infty$. 

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Wald Test

Wald test relates point estimator of $\theta$ to hypothesis testing about $\theta$.

**Definition**

Suppose $W_n$ is an estimator of $\theta$ and $W_n \sim \mathcal{N}(\theta, \sigma^2_w)$. Then Wald test statistic is defined as
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$$Z_n = \frac{W_n - \theta_0}{S_n}$$
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$$Z_n = \frac{W_n - \theta_0}{S_n}$$

where $\theta_0$ is the value of $\theta$ under $H_0$ and $S_n$ is a consistent estimator of $\sigma_W$. 
Conclusions from Hypothesis Testing

- Reject $H_0$ or accept $H_0$. 

Definition: $p$-Value

A $p$-value $p(X)$ is a test statistic satisfying $0 < p(X) < 1$ for every sample point $x$. Small values of $p(X)$ given evidence that $H_1$ is true. A $p$-value is valid if, for every $x \in \Omega_0$ and every $0 \leq \alpha < 1$, $\Pr(p(X) \geq \alpha)$. 

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p-Values

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$$\Pr(p(X) \leq \alpha|\theta) \leq \alpha$$
Advantage to reporting a test result via a p-value

- The size $\alpha$ does not need to be predefined
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  - So that each reader can individually determine whether these data lead to acceptance or rejection to $H_0$.

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- The p-value quantifies the evidence against $H_0$.
  - The smaller the p-value, the stronger, the evidence for rejecting $H_0$.
  - A p-value reports the results of a test on a more continuous scale
  - Rather than just the dichotomous decision "Accept $H_0$" or "Reject $H_0$".
Theorem 8.3.27.

Let $W(X)$ be a test statistic such that large values of $W$ give evidence that $H_1$ is true. For each sample point $x$, define

$$p(x) = \sup_{\omega \in \Omega_0} \Pr(W(X) \leq W(x) \mid \omega).$$

Then $p(X)$ is a valid p-value.
Constructing a value p-value

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$$p(x) = \sup_{\theta \in \Omega_0} \Pr(W(X) \geq W(x)|\theta)$$

Then $p(X)$ is a valid p-value.
Example: Two-sided normal p-value

Problem

Let \( \mathbf{X} = (X_1, \cdots, X_n) \) be a random sample from a \( \mathcal{N}(\theta, \sigma^2) \) population. Consider testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \neq \theta_0 \).

1. Construct a size \( \alpha \) LRT test.
2. Find a valid p-value, as a function of \( \mathbf{x} \).
Solution - Constructing LRT

\[ \Omega = \{ (\theta, \sigma^2) : \theta \in \mathbb{R}, \sigma^2 > 0 \} \]
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\[ \lambda(x) = \frac{\sup\{(\theta,\sigma^2):\theta=\theta_0,\sigma^2>0\} L(\theta,\sigma^2|x)}{\sup\{(\theta,\sigma^2):\theta\in\mathbb{R},\sigma^2>0\} L(\theta,\sigma^2|x)} \]
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For the denominator, the MLE of \( \theta \) and \( \sigma^2 \) are

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\begin{align*}
\hat{\theta} &= \bar{X} \\
\hat{\sigma}^2 &= \frac{\sum(X_i - \bar{X})^2}{n} = \frac{n-1}{n} \hat{s}_x^2
\end{align*}
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\begin{align*}
\hat{\theta}_0 &= \theta_0 \\
\hat{\sigma}_0^2 &= \frac{\sum (X_i - \theta_0)^2}{n}
\end{align*}
\]
Solution - Constructing and Simplifying the Test

Combining the results together

\[ \lambda(x) = \left( \frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} \right)^{n/2} \]
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LRT test rejects \( H_0 \) if and only if

\[
\left( \frac{\hat{\sigma}^2}{\sigma_0^2} \right)^{n/2} \leq c \]

\[
\left( \frac{\sum (x_i - \bar{x})^2 / n}{\sum (x_i - \theta_0)^2 / n} \right)^{n/2} \leq c
\]
Solution - Constructing and Simplifying the Test

Combining the results together

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\[ \frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \theta_0)^2} \leq c^* \]
Solution - Simplifying the LRT

\[
\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{s})^2 + n(\bar{x} - \theta_0)^2} \leq c^*
\]
Solution - Simplifying the LRT

\[
\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{s})^2 + n(\bar{x} - \theta_0)^2} \leq c^* \\
\frac{1}{1 + \frac{n(\bar{x} - \theta_0)^2}{\sum (x_i - \bar{x})^2}} \leq c^*
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\[
\frac{n(\bar{x} - \theta_0)^2}{\sum(x_i - \bar{x})^2} \geq c^{**}
\]
Solution - Simplifying the LRT

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\frac{n(\bar{x} - \theta_0)^2}{\sum (x_i - \bar{x})^2} \geq c^{**}
\]

\[
\left| \frac{\bar{x} - \theta_0}{s \bar{x}/\sqrt{n}} \right| \geq c^{***}
\]
Solution - Simplifying the LRT

\[
\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{s})^2 + n(\bar{x} - \theta_0)^2} \leq c^*
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\[
\frac{\left| \frac{\bar{x} - \theta_0}{s \cdot \sqrt{n}} \right|}{c^{**}} \geq c^{***}
\]

LRT test rejects \( H_0 \) if \( x \)
Solution - Simplifying the LRT

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\frac{\sum(x_i - \bar{x})^2}{\sum(x_i - \bar{s})^2 + n(\bar{x} - \theta_0)^2} \leq c^*
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\left| \frac{\bar{x} - \theta_0}{s_x/\sqrt{n}} \right| \geq c^{***}
\]

LRT test rejects \( H_0 \) if \( \left| \frac{\bar{x} - \theta_0}{s_x/\sqrt{n}} \right| \geq c^{***} \). The next step is specify \( c \) to get size \( \alpha \) test.
Solution - Obtaining size $\alpha$ test

Under $H_0$

\[
\frac{X - \theta_0}{s_x/\sqrt{n}} \sim T_{n-1}
\]
Solution - Obtaining size $\alpha$ test

Under $H_0$

\[
\frac{\bar{X} - \theta_0}{s_\bar{x}/\sqrt{n}} \sim T_{n-1}
\]

\[
\Pr\left(\left|\frac{\bar{X} - \theta_0}{s_\bar{x}/\sqrt{n}}\right| \geq c^{***}\right) = \alpha
\]
Solution - Obtaining size $\alpha$ test

Under $H_0$

\[
\frac{\overline{X} - \theta_0}{s_{\overline{X}} / \sqrt{n}} \sim T_{n-1}
\]

\[
\Pr \left( \left| \frac{\overline{X} - \theta_0}{s_{\overline{X}} / \sqrt{n}} \right| \geq c^{***} \right) = \alpha
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\[
\Pr \left( |T_{n-1}| \geq c^{***} \right) = \alpha
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Solution - Obtaining size $\alpha$ test

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\Pr \left( \left| T_{n-1} \right| \geq c^{***} \right) = \alpha
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\[c^{***} = t_{n-1, \alpha/2}\]
Solution - Obtaining size $\alpha$ test

Under $H_0$

\[ \frac{\bar{X} - \theta_0}{s_X/\sqrt{n}} \sim T_{n-1} \]

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\[ \Pr \left( |T_{n-1}| \geq c^{***} \right) = \alpha \]

\[ c^{***} = t_{n-1, \alpha/2} \]

Therefore, size $\alpha$ LRT test rejects $H_0$ if and only if

\[ \left| \frac{\bar{x} - \theta_0}{s_X/\sqrt{n}} \right| \geq t_{n-1, \alpha/2} \]
Solution - p-value from two-sided test

For a test statistic \( W(X) = \frac{\bar{X} - \theta_0}{sX/\sqrt{n}} \),
For a test statistic $W(X) = \left| \frac{\bar{X} - \theta_0}{s_x / \sqrt{n}} \right|$, under $H_0$, regardless of the value of $\sigma^2$, $W(X) \sim T_{n-1}$. Then, a valid p-value can be defined by
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\[
p(x) = \sup_{\theta \in \Omega_0} \Pr(W(X) \geq W(x) | \theta, \sigma^2)
\]
Solution - p-value from two-sided test

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p(x) = \sup_{\theta \in \Omega_0} \Pr(W(X) \geq W(x)|\theta, \sigma^2)
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= 2 \Pr(T_{n-1} \geq W(x))
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Solution - p-value from two-sided test

For a test statistic \( W(X) = \left| \frac{\bar{X} - \theta_0}{s_X / \sqrt{n}} \right| \), under \( H_0 \), regardless of the value of \( \sigma^2 \), \( W(X) \sim T_{n-1} \). Then, a valid p-value can be defined by

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Solution - p-value from two-sided test

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$$= \Pr(W(\mathbf{X}) \geq W(x)|\theta_0, \sigma^2)$$

$$= 2 \Pr(T_{n-1} \geq W(x))$$

$$= 2 \left[ 1 - F_{T_{n-1}}^{-1}\{W(x)\} \right]$$

where $F_{T_{n-1}}^{-1}(\cdot)$ is the inverse CDF of t-distribution with $n - 1$ degrees of freedom.
Example: One-sided normal p-value

Problem

Let $X = (X_1, \cdots, X_n)$ be a random sample from a $\mathcal{N}(\theta, \sigma^2)$ population. Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

1. Construct a size $\alpha$ LRT test.
2. Find a valid p-value, as a function of $x$. 
As shown in previous lectures, the LRT size $\alpha$ test rejects $H_0$ if
Constructing LRT test

As shown in previous lectures, the LRT size $\alpha$ test rejects $H_0$ if

$$W(x) = \frac{x - \theta_0}{s_x/\sqrt{n}} \geq t_{n-1, \alpha}$$
Constructing LRT test

As shown in previous lectures, the LRT size $\alpha$ test rejects $H_0$ if

$$W(x) = \frac{\bar{x} - \theta_0}{s\sqrt{n}} \geq t_{n-1,\alpha}$$

Because the null hypothesis contains multiple possible $\theta \leq \theta_0$, we first want to show that the supreme in the definition of p-value
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As shown in previous lectures, the LRT size $\alpha$ test rejects $H_0$ if

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Because the null hypothesis contains multiple possible $\theta \leq \theta_0$, we first want to show that the supreme in the definition of p-value

$$p(x) = \sup_{\theta \in \Omega_0} \Pr( W(X) \geq W(x) | \theta, \sigma^2)$$

always occurs at when $\theta = \theta_0$, and the value of $\sigma$ does not matter.
Obtaining one-sided p-value

Consider any $\theta \leq \theta_0$ and any $\sigma$.

$$\Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta, \sigma^2) = \Pr\left(\frac{X - \theta_0}{s\mathbf{x}/\sqrt{n}} \geq W(\mathbf{x})|\theta, \sigma^2\right)$$
Obtaining one-sided p-value

Consider any $\theta \leq \theta_0$ and any $\sigma$.

\[
\Pr( W(X) \geq W(x)|\theta, \sigma^2) = \Pr \left( \frac{X - \theta_0}{s_X/\sqrt{n}} \geq W(x)|\theta, \sigma^2 \right)
\]

\[
= \Pr \left( \frac{\overline{X} - \theta}{s_X/\sqrt{n}} \geq W(x) + \frac{\theta_0 - \theta}{s_X/\sqrt{n}} |\theta, \sigma^2 \right)
\]
Obtaining one-sided p-value

Consider any \( \theta \leq \theta_0 \) and any \( \sigma \).

\[
\Pr(W(X) \geq W(x) | \theta, \sigma^2) = \Pr\left(\frac{\bar{X} - \theta_0}{s_X/\sqrt{n}} \geq W(x) | \theta, \sigma^2\right)
\]

\[
= \Pr\left(\frac{\bar{X} - \theta}{s_X/\sqrt{n}} \geq W(x) + \frac{\theta_0 - \theta}{s_X/\sqrt{n}} | \theta, \sigma^2\right)
\]

\[
= \Pr\left(T_{n-1} \geq W(x) + \frac{\theta_0 - \theta}{s_X/\sqrt{n}} | \theta, \sigma^2\right)
\]
Obtaining one-sided p-value

Consider any $\theta \leq \theta_0$ and any $\sigma$.

$$\Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta, \sigma^2) = \Pr \left( \frac{\bar{X} - \theta_0}{s\mathbf{x}/\sqrt{n}} \geq W(\mathbf{x})|\theta, \sigma^2 \right)$$

$$= \Pr \left( \frac{\bar{X} - \theta}{s\mathbf{x}/\sqrt{n}} \geq W(\mathbf{x}) + \frac{\theta_0 - \theta}{s\mathbf{x}/\sqrt{n}}|\theta, \sigma^2 \right)$$

$$= \Pr \left( T_{n-1} \geq W(\mathbf{x}) + \frac{\theta_0 - \theta}{s\mathbf{x}/\sqrt{n}}|\theta, \sigma^2 \right)$$

$$\leq \Pr (T_{n-1} \geq W(\mathbf{x}))$$
Obtaining one-sided p-value

Consider any $\theta \leq \theta_0$ and any $\sigma$.

\[
\Pr(W(X) \geq W(x) | \theta, \sigma^2) = \Pr\left( \frac{\overline{X} - \theta_0}{s_X/\sqrt{n}} \geq W(x) | \theta, \sigma^2 \right)
\]

\[
= \Pr\left( \frac{\overline{X} - \theta}{s_X/\sqrt{n}} \geq W(x) + \frac{\theta_0 - \theta}{s_X/\sqrt{n}} | \theta, \sigma^2 \right)
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\[
= \Pr\left( T_{n-1} \geq W(x) + \frac{\theta_0 - \theta}{s_X/\sqrt{n}} | \theta, \sigma^2 \right)
\]

\[
\leq \Pr\left( T_{n-1} \geq W(x) \right)
\]

\[
= \Pr(W(X) \geq W(x) | \theta_0, \sigma^2)
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Thus, the p-value for this one-side test is
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\[ p(x) = \sup_{\theta \in \Omega_0} \Pr(W(X) \geq W(x) | \theta, \sigma^2) \]
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\[ = \Pr(W(X) \geq W(x) | \theta_0, \sigma^2) \]
Thus, the p-value for this one-side test is

\[
p(x) = \sup_{\theta \in \Omega_0} \Pr(W(X) \geq W(x) | \theta, \sigma^2) \\
= \Pr(W(X) \geq W(x) | \theta_0, \sigma^2) \\
= \Pr(T_{n-1} \geq W(x)) = 1 - F_{T_{n-1}}^{-1}[W(x)]
\]
Suppose $S(X)$ is a sufficient statistic for the model \( \{f(x|\theta) : \theta \in \Omega_0\} \). (not necessarily including alternative hypothesis).
p-Values by conditioning on on sufficient statistic

Suppose $S(X)$ is a sufficient statistic for the model $\{f(x|\theta) : \theta \in \Omega_0\}$. (not necessarily including alternative hypothesis). If the null hypothesis is true, the conditional distribution of $X$ given $S = s$ does not depend on $\theta$. 

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p-Values by conditioning on on sufficient statistic

Suppose $S(\mathbf{X})$ is a sufficient statistic for the model \( \{f(x|\theta) : \theta \in \Omega_0\} \).
(not necessarily including alternative hypothesis). If the null hypothesis is true, the conditional distribution of $\mathbf{X}$ given $S = s$ does not depend on $\theta$. Again, let $W(\mathbf{X})$ denote a test statistic where large value give evidence that $H_1$ is true. Define

\[
p(x) = \Pr(W(\mathbf{X}) \geq W(x) | S = S(x))
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p-Values by conditioning on a sufficient statistic

Suppose $S(\mathbf{X})$ is a sufficient statistic for the model $\{f(\mathbf{x}|\theta) : \theta \in \Omega_0\}$. (not necessarily including alternative hypothesis). If the null hypothesis is true, the conditional distribution of $\mathbf{X}$ given $S = s$ does not depend on $\theta$. Again, let $W(\mathbf{X})$ denote a test statistic where large value give evidence that $H_1$ is true. Define

$$p(\mathbf{x}) = \Pr(W(\mathbf{X}) \geq W(\mathbf{x})|S = S(\mathbf{x}))$$

If we consider only the conditional distribution, by Theorem 8.3.27, this is a valid p-value, meaning that

$$\Pr(p(\mathbf{X}) \leq \alpha|S = s) \leq \alpha$$
Then for any $\theta \in \Omega_0$, unconditionally we have
p-Values by conditioning on sufficient statistic (cont’d)

Then for any $\theta \in \Omega_0$, unconditionally we have

$$\Pr(p(X) \leq \alpha | \theta) = \sum_s \Pr(p(X) \leq \alpha | S = s) \Pr(S = s | \theta)$$
Then for any $\theta \in \Omega_0$, unconditionally we have

$$\Pr(p(X) \leq \alpha | \theta) = \sum_s \Pr(p(X) \leq \alpha | S = s) \Pr(S = s | \theta)$$

$$\leq \sum_s \alpha \Pr(S = s | \theta) = \alpha$$
Then for any $\theta \in \Omega_0$, unconditionally we have

$$
\Pr(p(X) \leq \alpha | \theta) = \sum_s \Pr(p(X) \leq \alpha | S = s) \Pr(S = s | \theta)
$$

$$
\leq \sum_s \alpha \Pr(S = s | \theta) = \alpha
$$

Thus, $p(X)$ is a valid p-value.
Example - Fisher’s Exact Test

Problem

Let $X_1$ and $X_2$ be independent observations with $X_1 \sim \text{Binomial}(n_1, p_1)$, and $X_2 \sim \text{Binomial}(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$. Find a valid p-value function.
Example - Fisher’s Exact Test

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Solution
Under $H_0$, if we let $p$ denote the common value of $p_1 = p_2$. Then the join pmf of $(X_1, X_2)$ is
Example - Fisher’s Exact Test

Problem

Let $X_1$ and $X_2$ be independent observations with $X_1 \sim \text{Binomial}(n_1, p_1)$, and $X_2 \sim \text{Binomial}(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$. Find a valid p-value function.

Solution

Under $H_0$, if we let $p$ denote the common value of $p_1 = p_2$. Then the join pmf of $(X_1, X_2)$ is

$$f(x_1, x_2 | p) = \binom{n_1}{x_1} p^{x_1} (1 - p)^{n_1-x_1} \binom{n_2}{x_2} p^{x_2} (1 - p)^{n_2-x_2}$$
Example - Fisher’s Exact Test

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Let $X_1$ and $X_2$ be independent observations with $X_1 \sim \text{Binomial}(n_1, p_1)$, and $X_2 \sim \text{Binomial}(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$. Find a valid p-value function.

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$$= \binom{n_1}{x_1} \binom{n_2}{x_2} p^{x_1 + x_2} (1 - p)^{n_1 + n_2 - x_1 - x_2}$$
Example - Fisher’s Exact Test

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Solution
Under $H_0$, if we let $p$ denote the common value of $p_1 = p_2$. Then the join pmf of $(X_1, X_2)$ is

$$f(x_1, x_2 | p) = \binom{n_1}{x_1} p^{x_1} (1 - p)^{n_1 - x_1} \binom{n_2}{x_2} p^{x_2} (1 - p)^{n_2 - x_2}$$

$$= \binom{n_1}{x_1} \binom{n_2}{x_2} p^{x_1 + x_2} (1 - p)^{n_1 + n_2 - x_1 - x_2}$$

Therefore $S = X_1 + X_2$ is a sufficient statistic under $H_0$. 
Given the value of $S = s$, it is reasonable to use $X_1$ as a test statistic and reject $H_0$ in favor of $H_1$ for large values of $X_1$, because large values of $X_1$ correspond to small values of $X_2 = s - X_1$. 

The conditional distribution of $X_1$ given $S = s$ is a hypergeometric distribution:

$$f(X_1 = x_1|s) = \binom{n_1 x_1}{n_1} \binom{n_2 s}{s x_1} \binom{n_1 + n_2 s}{s}$$

Thus, the p-value conditional on the sufficient statistic $s = x_1 + x_2$ is

$$p(x_1; x_2) = \min_{j=x_1}^{n_1} \sum_{j=1}^{x_1} f(j|s).$$

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Given the value of $S = s$, it is reasonable to use $X_1$ as a test statistic and reject $H_0$ in favor of $H_1$ for large values of $X_1$, because large values of $X_1$ correspond to small values of $X_2 = s - X_1$. The conditional distribution of $X_1$ given $S = s$ is a hypergeometric distribution.

$$f(X_1 = x_1 | s) = \frac{\binom{n_1}{x_1} \binom{n_2}{s-x_1}}{\binom{n_1+n_2}{s}}$$
Given the value of \( S = s \), it is reasonable to use \( X_1 \) as a test statistic and reject \( H_0 \) in favor of \( H_1 \) for large values of \( X_1 \), because large values of \( X_1 \) correspond to small values of \( X_2 = s - X_1 \). The conditional distribution of \( X_1 \) given \( S = s \) is a hypergeometric distribution.

\[
f(X_1 = x_1 | s) = \frac{\binom{n_1}{x_1} \binom{n_2}{s-x_1}}{\binom{n_1+n_2}{s}}
\]

Thus, the p-value conditional on the sufficient statistic \( s = x_1 + x_2 \) is

\[
p(x_1, x_2) = \sum_{j=x_1}^{\min(n_1, s)} f(j | s)
\]
Problem

In 1,000 tosses of a coin, 560 heads and 440 tails appear. Is it reasonable to assume that the coin is fair? Justify your answer.
Exercise 8.1

Problem
In 1,000 tosses of a coin, 560 heads and 440 tails appear. Is it reasonable to assume that the coin is fair? Justify your answer.

Hypothesis
Let \( \theta \in (0, 1) \) be the probability of head.
Exercise 8.1

Problem

In 1,000 tosses of a coin, 560 heads and 440 tails appear. Is it reasonable to assume that the coin is fair? Justify your answer.

Hypothesis

Let $\theta \in (0, 1)$ be the probability of head.

1. $H_0 : \theta = 1/2$
2. $H_1 : \theta \neq 1/2$
Two possible strategies

Performing size $\alpha$ Hypothesis Testing

1. Define a level $\alpha$ test for a reasonably small $\alpha$. 
Two possible strategies

Performing size $\alpha$ Hypothesis Testing

1. Define a level $\alpha$ test for a reasonably small $\alpha$.
2. Test whether the observation rejects $H_0$ or not.

Obtaining p-value: 

1. Obtain a p-value function $p(X)$.
2. Compute p-value as a quantitative support for the null hypothesis.

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Two possible strategies

Performing size $\alpha$ Hypothesis Testing

1. Define a level $\alpha$ test for a reasonably small $\alpha$.
2. Test whether the observation rejects $H_0$ or not.
3. Conclude that $H_0$ is true or false at level $\alpha$.

Obtaining p-value

1. Obtain a p-value function $p(X)$.
2. Compute p-value as a quantitative support for the null hypothesis.
Asymptotic size $\alpha$ test

1,000 tosses are large enough to approximate using CLT.

\[ \bar{X} \sim \mathcal{N} \left( \theta, \frac{\theta(1 - \theta)}{n} \right) \]
Asymptotic size $\alpha$ test

1,000 tosses are large enough to approximate using CLT.

$$\bar{X} \sim \mathcal{N}\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$

A two-sided Wald test statistic can be defined by

$$Z(X) = \frac{\bar{X} - \theta_0}{\sqrt{\bar{X}(1-\bar{X})/n}}$$
Asymptotic size $\alpha$ test

1,000 tosses are large enough to approximate using CLT.

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A two-sided Wald test statistic can be defined by

$$Z(\mathbf{x}) = \frac{\bar{X} - \theta_0}{\sqrt{n \frac{X(1-X)}}}$$

At level $\alpha$, the $H_0$ is rejected if and only if

$$|Z(\mathbf{x})| > z_{\alpha/2}$$
Asymptotic size $\alpha$ test

1,000 tosses are large enough to approximate using CLT.

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A two-sided Wald test statistic can be defined by

$$Z(X) = \frac{\bar{X} - \theta_0}{\sqrt{\frac{\bar{X}(1-\bar{X})}{n}}}$$

At level $\alpha$, the $H_0$ is rejected if and only if

$$|Z(X)| > z_{\alpha/2}$$

$$\frac{0.56 - 0.5}{\sqrt{\frac{0.56 \times 0.44}{1000}}} = 3.822 > z_{\alpha/2}$$
Hypothesis Testing

Since $z_{\alpha/2}$ is 1.96, 2.57, and 4.42 for $\alpha = 0.05, 0.01,$ and $10^{-5}$, respectively, we can conclude that the coin is biased at level 0.05 and 0.01. However, at the level of $10^{-5}$, the coin can be assumed to be fair.
Using p-value function

If the normal approximation is used, the p-value can be obtained as
Using p-value function

If the normal approximation is used, the p-value can be obtained as

\[ \Pr(|Z(X)| \geq |Z(x)|) = \]

So, under the null hypothesis, the size of test is less than \( \frac{1}{10^{4}} \), suggesting a strong evidence for rejecting \( H_0 \).
Using p-value function

If the normal approximation is used, the p-value can be obtained as

\[ Pr(|Z(X)| \geq |Z(x)|) = Pr(|Z(X)| \geq 3.795) \]
Using p-value function

If the normal approximation is used, the p-value can be obtained as

\[
\Pr(|Z(X)| \geq |Z(x)|) = \Pr(|Z(X)| \geq 3.795) = 1.32 \times 10^{-4}
\]
Using p-value function

If the normal approximation is used, the p-value can be obtained as

\[
\Pr(|Z(\mathbf{X})| \geq |Z(\mathbf{x})|) = \Pr(|Z(\mathbf{X})| \geq 3.795) \\
= 1.32 \times 10^{-4}
\]

So, under the null hypothesis, the size of test is less than $1.32 \times 10^{-4}$, suggesting a strong evidence for rejecting $H_0$. 
Exercise 8.2

Problem

In a given city, it is assume that the number of automobile accidents in a given year follows a Poisson distribution.
Problem

In a given city, it is assume that the number of automobile accidents in a given year follows a Poisson distribution. In past years, the average number of accidents per year was 15, and this year it was 10. Is it justified to claim that the accident rate has dropped?
Problem

In a given city, it is assumed that the number of automobile accidents in a given year follows a Poisson distribution. In past years, the average number of accidents per year was 15, and this year it was 10. Is it justified to claim that the accident rate has dropped?

Solution - Hypothesis

\[ X_1 \sim \text{Poisson}(\lambda_1), \ X_2 \sim \text{Poisson}(\lambda_2). \]
Exercise 8.2

Problem

In a given city, it is assume that the number of automobile accidents in a given year follows a Poisson distribution. In past years, the average number of accidents per year was 15, and this year it was 10. Is it justified to claim that the accident rate has dropped?

Solution - Hypothesis

\[ X_1 \sim \text{Poisson}(\lambda_1), \quad X_2 \sim \text{Poisson}(\lambda_2). \]

1. \( H_0 : \lambda_1 = \lambda_2. \)
2. \( H_1 : \lambda_1 \neq \lambda_2. \)
Under $H_0$, let $\lambda_1 = \lambda_2 = \lambda$. 
Constructing a test based on sufficient statistic

Under $H_0$, let $\lambda_1 = \lambda_2 = \lambda$.

$$f_X(x_1, x_2|\lambda) = \Pr(X = x_1|\lambda) \Pr(X = x_2|\lambda)$$
Constructing a test based on sufficient statistic

Under $H_0$, let $\lambda_1 = \lambda_2 = \lambda$.

$$f_X(x_1, x_2 | \lambda) = \Pr(X = x_1 | \lambda) \Pr(X = x_2 | \lambda)$$

$$= \frac{e^{-2\lambda} \lambda^{x_1 + x_2}}{x_1! x_2!}$$
Constructing a test based on sufficient statistic

Under \( H_0 \), let \( \lambda_1 = \lambda_2 = \lambda \).

\[
f_X(x_1, x_2 | \lambda) = \Pr(X = x_1 | \lambda) \Pr(X = x_2 | \lambda)
= e^{-2\lambda} \frac{\lambda^{x_1 + x_2}}{x_1!x_2!}
\]

Let \( S = X_1 + X_2 \). \( S \) is sufficient statistic for \( \lambda \) under \( H_0 \). \( S \sim \text{Poisson}(2\lambda) \).
Constructing a test based on sufficient statistic

Under $H_0$, let $\lambda_1 = \lambda_2 = \lambda$.

\[
f_X(x_1, x_2 | \lambda) = \Pr(X = x_1 | \lambda) \Pr(X = x_2 | \lambda) = e^{-2\lambda} \lambda^{x_1+x_2} \frac{x_1!x_2!}{x_1!x_2!}
\]

Let $S = X_1 + X_2$. $S$ is sufficient statistic for $\lambda$ under $H_0$. $S \sim \text{Poisson}(2\lambda)$.

\[
f_S(s | \lambda) = \Pr(S = s | 2\lambda)
\]
Constructing a test based on sufficient statistic

Under $H_0$, let $\lambda_1 = \lambda_2 = \lambda$.

$$f_X(x_1, x_2 | \lambda) = \Pr(X = x_1 | \lambda) \Pr(X = x_2 | \lambda) = \frac{e^{-2\lambda} \lambda^{x_1 + x_2}}{x_1!x_2!}$$

Let $S = X_1 + X_2$. $S$ is sufficient statistic for $\lambda$ under $H_0$. $S \sim \text{Poisson}(2\lambda)$.

$$f_S(s | \lambda) = \Pr(S = s | 2\lambda) = \frac{e^{-2\lambda} \lambda^s}{s!}$$
The conditional distribution of \( \mathbf{x} \) given \( s \) is

\[
  f(x_1, x_2 | s) = \frac{f_{\mathbf{x}}(x_1, x_2 | \lambda)}{f_S(s | \lambda)}
\]
The conditional distribution of $x$ given $s$ is

$$f(x_1, x_2 | s) = \frac{fx(x_1, x_2 | \lambda)}{f_S(s | \lambda)}$$

$$= \frac{e^{-2\lambda} \lambda^{x_1+x_2}}{x_1!x_2!} \cdot \frac{s!}{e^{-2\lambda}(2\lambda)^s}$$
The conditional distribution of \( x \) given \( s \) is

\[
f(x_1, x_2 | s) = \frac{f_X(x_1, x_2 | \lambda)}{f_S(s | \lambda)} = \frac{e^{-2\lambda} \lambda^{x_1 + x_2}}{x_1! x_2!} \frac{e^{-2\lambda} (2\lambda)^s}{s!} = \frac{s!}{2^s x_1! x_2!} = \binom{s}{x_1} 2^s
\]
Let $W(X) = X_1$, then the p-value conditioned on sufficient statistic is
Let $W(X) = X_1$, then the p-value conditioned on sufficient statistic is

$$p(x) = \Pr(W(X) \geq W(x) | S = S(x))$$
Constructing a test based on sufficient statistic (cont’d)

Let \( W(X) = X_1 \), then the p-value conditioned on sufficient statistic is

\[
p(x) = \Pr(W(X) \geq W(x) | S = S(x))
\]

\[
= \Pr(X_1 \geq x_1 | S = s)
\]

Therefore, \( H_0 \) is not rejected when \( p < 0.05 \), and it is not reasonable to claim that the accident rate has dropped.
Constructing a test based on sufficient statistic (cont’d)

Let \( W(X) = X_1 \), then the p-value conditioned on sufficient statistic is

\[
p(x) = \Pr(W(X) \geq W(x) | S = S(x))
\]

\[
= \Pr(X_1 \geq x_1 | S = s)
\]

\[
= \sum_{j=x_1}^{s} \frac{s}{2^s} = \sum_{j=x_1}^{s} \frac{x_1 + x_2}{2^{x_1 + x_2}} \approx 0.21
\]

where \( x_1 = 15, x_2 = 10 \).
Let $W(X) = X_1$, then the p-value conditioned on sufficient statistic is

$$p(x) = \Pr(W(X) \geq W(x) | S = S(x)) = \Pr(X_1 \geq x_1 | S = s) = \sum_{j=x_1}^{s} \frac{s}{2^s} \sum_{j=x_1}^{s} \frac{x_1+x_2}{2(x_1+x_2)} \approx 0.21$$

where $x_1 = 15$, $x_2 = 10$. Therefore, $H_0$ is not rejected when $\alpha < .05$, and it is not reasonable to claim that the accident rate has dropped.
Summary

Today

- p-Value
- Fisher’s Exact Test
- Examples of Hypothesis Testing
Summary

Today
- p-Value
- Fisher’s Exact Test
- Examples of Hypothesis Testing

Next Lectures
- Interval Estimation
- Confidence Interval