Biostatistics 602 - Statistical Inference
Lecture 19
Likelihood Ratio Test

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Last Lecture

Describe the following concepts in your own words

- Hypothesis
- Null Hypothesis
- Alternative Hypothesis
- Hypothesis Testing Procedure
- Rejection Region
- Type I error
- Type II error
- Power function
- Size $\alpha$ test
- Level $\alpha$ test
- Likelihood Ratio Test

Example of Hypothesis Testing

Let $X_1, \cdots, X_n$ be changes in blood pressure after a treatment.

$H_0 : \theta = 0$

$H_1 : \theta \neq 0$

The rejection region $= \{x : \frac{\bar{x}}{s_x/\sqrt{n}} > 3\}$.

<table>
<thead>
<tr>
<th>Truth</th>
<th>Accept $H_0$</th>
<th>Reject $H_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0$</td>
<td>Correct Decision</td>
<td>Type I error</td>
</tr>
<tr>
<td>$H_1$</td>
<td>Type II error</td>
<td>Correct Decision</td>
</tr>
</tbody>
</table>

Power function

Definition - The power function

The power function of a hypothesis test with rejection region $R$ is the function of $\theta$ defined by

$$\beta(\theta) = \Pr(X \in R|\theta) = \Pr(\text{reject } H_0|\theta)$$

If $\theta \in \Omega_0^c$ (alternative is true), the probability of rejecting $H_0$ is called the power of test for this particular value of $\theta$.

- Probability of type I error $= \beta(\theta)$ if $\theta \in \Omega_0$.
- Probability of type II error $= 1 - \beta(\theta)$ if $\theta \in \Omega_0^c$.

An ideal test should have power function satisfying $\beta(\theta) = 0$ for all $\theta \in \Omega_0$, $\beta(\theta) = 1$ for all $\theta \in \Omega_0^c$, which is typically not possible in practice.
Sizes and Levels of Tests

Size $\alpha$ test

A test with power function $\beta(\theta)$ is a size $\alpha$ test if
$$\sup_{\theta \in \Omega_0} \beta(\theta) = \alpha$$

In other words, the maximum probability of making a type I error is $\alpha$.

Level $\alpha$ test

A test with power function $\beta(\theta)$ is a level $\alpha$ test if
$$\sup_{\theta \in \Omega_0} \beta(\theta) \leq \alpha$$

In other words, the maximum probability of making a type I error is equal or less than $\alpha$.

Any size $\alpha$ test is also a level $\alpha$ test.

Likelihood Ratio Tests (LRT)

Definition

Let $L(\theta|x)$ be the likelihood function of $\theta$. The likelihood ratio test statistic for testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_0^1$ is
$$\lambda(x) = \frac{\sup_{\theta \in \Omega_0} L(\theta|x)}{\sup_{\theta \in \Omega} L(\theta|x)} = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)}$$

where $\hat{\theta}$ is the MLE of $\theta$ over $\Omega$, and $\hat{\theta}_0$ is the MLE of $\theta$ over $\theta \in \Omega_0$ (restricted MLE).

The likelihood ratio test is a test that rejects $H_0$ if and only if $\lambda(x) \leq c$ where $0 \leq c \leq 1$.

Example of LRT

Problem

Consider $X_1, \cdots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2)$ where $\sigma^2$ is known.

$H_0 : \theta \leq \theta_0$

$H_1 : \theta > \theta_0$

For the LRT test and its power function

Solution

$$L(\theta|x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i - \theta)^2}{2\sigma^2}\right]$$

We need to find MLE of $\theta$ over $\Omega = (-\infty, \infty)$ and $\Omega_0 = (-\infty, \theta_0]$.
MLE of $\theta$ over $\Omega_0 = (-\infty, \theta_0]$  

- $L(\theta|x)$ is maximized at $\theta = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$ if $\bar{x} \leq \theta_0$.
- However, if $\bar{x} \geq \theta_0$, $\bar{x}$ does not fall into a valid range of $\hat{\theta}_0$, and $\theta \leq \theta_0$, the likelihood function will be an increasing function. Therefore $\hat{\theta}_0 = \theta_0$.

To summarize,

$$\hat{\theta}_0 = \begin{cases} \bar{x} & \text{if } \bar{x} \leq \theta_0 \\ \theta_0 & \text{if } \bar{x} > \theta_0 \end{cases}$$

Likelihood ratio test

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} = \begin{cases} 1 & \text{if } \bar{X} \leq \theta_0 \\ \exp \left[ -\frac{\sum_{i=1}^{n} (x_i - \theta_0)^2}{2\sigma^2} \right] & \text{if } \bar{X} > \theta_0 \end{cases}$$

Therefore, the likelihood test rejects the null hypothesis if and only if

$$\exp \left[ -\frac{n(\bar{x} - \theta_0)^2}{2\sigma^2} \right] \leq c$$

and $\bar{x} \geq \theta_0$.

Specifying $c$

$$\exp \left[ -\frac{n(\bar{x} - \theta_0)^2}{2\sigma^2} \right] \leq c$$

$$\iff -\frac{n(\bar{x} - \theta_0)^2}{2\sigma^2} \leq \log c$$

$$\iff (\bar{x} - \theta_0)^2 \geq -\frac{2\sigma^2 \log c}{n}$$

$$\iff \bar{x} - \theta_0 \geq \sqrt{-\frac{2\sigma^2 \log c}{n}} \quad (\because \bar{x} > \theta_0)$$

Specifying $c$ (cont’d)

So, LRT rejects $H_0$ if and only if

$$\bar{x} - \theta_0 \geq \sqrt{-\frac{2\sigma^2 \log c}{n}}$$

$$\iff \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq \sqrt{-\frac{2\sigma^2 \log c}{n}} = c^*$$

Therefore, the rejection region is

$$\left\{ x : \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq c^* \right\}$$
**Power function**

\[ \beta(\theta) = \Pr(\text{reject } H_0) = \Pr(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} \geq c^*) \]

\[ = \Pr(\frac{\bar{X} - \theta + \theta - \theta_0}{\sigma/\sqrt{n}} \geq c^*) \]

\[ = \Pr(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*) \]

Since \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2) \), \( \bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right) \). Therefore,

\[ \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \sim N(0, 1) \]

\[ \implies \beta(\theta) = \Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right) \]

where \( Z \sim N(0, 1) \).

**Making size \( \alpha \) LRT**

To make a size \( \alpha \) test,

\[ \sup_{\theta \in \Omega_0} \beta(\theta) = \alpha \]

\[ \sup_{\theta \leq \theta_0} \Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right) = \alpha \]

\[ \Pr(Z \geq c^*) = \alpha \]

\[ c^* = z_\alpha \]

Note that \( \Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right) \) is maximized when \( \theta \) is maximum (i.e. \( \theta = \theta_0 \)).

Therefore, size \( \alpha \) LRT test rejects \( H_0 \) if and only if \( \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq z_\alpha \).

**Another Example of LRT**

**Problem**

\( X_1, \ldots, X_n \overset{i.i.d.}{\sim} f(x|\theta) = e^{-(x-\theta)} \) where \( x \geq \theta \) and \(-\infty < \theta < \infty \). Find a LRT testing the following one-sided hypothesis.

\[ H_0 : \theta \leq \theta_0 \]

\[ H_1 : \theta > \theta_0 \]

**Solution**

\[ L(\theta|x) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I(x_i \geq \theta) \]

\[ = e^{-\sum x_i + n\theta} I(\theta \leq x(1)) \]

The likelihood function is a increasing function of \( \theta \), bounded by \( \theta \leq x(1) \). Therefore, when \( \theta \in \Omega = \mathbb{R} \), \( L(\theta|x) \) is maximized when \( \theta = \hat{\theta} = x(1) \).
Solution (cont’d)

The LRT rejects $H_0$ if and only if

\[ e^{n(\theta_0 - x(1))} \leq c \quad \text{(and } \theta_0 < x(1)) \]

\[ \iff \theta_0 - x(1) \leq \frac{\log c}{n} \]

\[ \iff x(1) \geq \theta_0 - \frac{\log c}{n} \]

So, LRT reject $H_0$ if $x(1) \geq \theta_0 - \frac{\log c}{n}$ and $x(1) > \theta_0$. The power function is

\[ \beta(\theta) = \Pr \left( X(1) \leq \theta_0 - \frac{\log c}{n} \land X(1) > \theta_0 \right) \]

To find size $\alpha$ test, we need to find $c$ satisfying the condition

\[ \sup_{\theta \leq \theta_0} \beta(\theta) = \alpha \]

Proof

By Factorization Theorem, the joint pdf of $x$ can be written as

\[ f(x|\theta) = g(T(x)|\theta)h(x) \]

and we can choose $g(t|\theta)$ to be the pdf or pmf of $T(x)$. Then, the LRT statistic based on $T(X)$ is defined as

\[ \lambda^*(t) = \sup_{\theta \in \Omega_0} \frac{L(\theta|T(x) = t)}{\sup_{\theta \in \Omega} L(\theta|T(x) = t)} = \sup_{\theta \in \Omega_0} \frac{g(t|\theta)}{\sup_{\theta \in \Omega} g(t|\theta)} \]

LRT statistic based on $X$ is

\[ \lambda(x) = \frac{\sup_{\theta \in \Omega_0} L(\theta|x)}{\sup_{\theta \in \Omega} L(\theta|x)} = \frac{\sup_{\theta \in \Omega_0} f(x|\theta)}{\sup_{\theta \in \Omega} f(x|\theta)} \]

Theorem 8.2.4

If $T(X)$ is a sufficient statistic for $\theta$, $\lambda^*(t)$ is the LRT statistic based on $T$, and $\lambda(x)$ is the LRT statistic based on $x$ then

\[ \lambda^*[T(X)] = \lambda(x) \]

for every $x$ in the sample space.

Proof (cont’d)

LRT statistic based on $X$ is

\[ \lambda(x) = \frac{\sup_{\theta \in \Omega_0} L(\theta|x)}{\sup_{\theta \in \Omega} L(\theta|x)} = \frac{\sup_{\theta \in \Omega_0} f(x|\theta)}{\sup_{\theta \in \Omega} f(x|\theta)} \]

The simplified expression of $\lambda(x)$ should depend on $x$ only through $T(x)$, where $T(x)$ is a sufficient statistic for $\theta$. 
Example
Problem
Consider \( X_1, \cdots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2) \) where \( \sigma^2 \) is known.
\[
H_0 : \theta = \theta_0 \\
H_1 : \theta \neq \theta_0
\]
Find a size \( \alpha \) LRT.

Solution - Using sufficient statistics
\( T(X) = \bar{X} \) is a sufficient statistic for \( \theta \).
\[
T \sim N\left(\theta, \frac{\sigma^2}{n}\right)
\]
\[
\lambda(t) = \frac{\sup_{\theta \in \Omega_0} L(\theta|t)}{\sup_{\theta \in \Omega} L(\theta|t)} = \exp\left[\frac{(t-\theta_0)^2}{2\sigma^2/n}\right]
\]
\[
\Rightarrow \left| \frac{T - \theta_0}{\sigma/\sqrt{n}} \right| \geq \sqrt{-2 \log c} = c^*
\]

Solution (cont’d)
Note that
\[
T = \bar{X} \sim N\left(\theta, \frac{\sigma^2}{n}\right)
\]
\[
\frac{T - \theta_0}{\sigma/\sqrt{n}} \sim N(0, 1)
\]
A size \( \alpha \) test satisfies
\[
\sup_{\theta \in \Omega_0} \Pr\left(\left| \frac{T - \theta}{\sigma/\sqrt{n}} \right| \geq c^* \right) = \alpha
\]
\[
\Pr\left(\left| \frac{T - \theta_0}{\sigma/\sqrt{n}} \right| \geq c^* \right) = \alpha
\]
\[
\Pr(|Z| \geq c^*) = \alpha
\]
\[
\Pr(Z \geq c^*) + \Pr(Z \leq -c^*) = \alpha
\]
\[
|Z| = \left| \frac{T - \theta}{\sigma/\sqrt{n}} \right| \geq z_{\alpha/2}
\]

Solution (cont’d)
The numerator is fixed, and MLE in the denominator is \( \hat{\theta} = t \). Therefore the LRT statistic is
\[
\lambda(t) = \exp\left[\frac{-n(t - \theta_0)^2}{2\sigma^2}\right]
\]
LRT rejects \( H_0 \) if and only if
\[
\lambda(t) = \exp\left[\frac{-n(t - \theta_0)^2}{2\sigma^2}\right] \leq c
\]
\[
\Rightarrow \left| \frac{t - \theta_0}{\sigma/\sqrt{n}} \right| \geq \sqrt{-2 \log c} = c^*
\]

LRT with nuisance parameters
Problem
\( X_1, \cdots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2) \) where both \( \theta \) and \( \sigma^2 \) unknown. Between \( H_0 : \theta \leq \theta_0 \) and \( H_1 : \theta > \theta_0 \).

1. Specify \( \Omega \) and \( \Omega_0 \)
2. Find size \( \alpha \) LRT.

Solution - \( \Omega \) and \( \Omega_0 \)
\[
\Omega = \{(\theta, \sigma^2) : \theta \in \mathbb{R}, \sigma^2 > 0\}
\]
\[
\Omega_0 = \{(\theta, \sigma^2) : \theta \leq \theta_0, \sigma^2 > 0\}
\]
Solution - Size $\alpha$ LRT

$$\lambda(x) = \sup_{\{\theta, \sigma^2: \theta \leq \theta_0, \sigma^2 > 0\}} L(\theta, \sigma^2|x)$$

For the denominator, the MLE of $\theta$ and $\sigma^2$ are

$$\hat{\theta} = \bar{x}$$
$$\hat{\sigma}^2 = \frac{\sum (x_i - \bar{x})^2}{n} = \frac{n-1}{n} \bar{x}^2$$

For numerator, we need to maximize $L(\theta, \sigma^2|x)$ over the region $\theta \leq \theta_0$ and $\sigma^2 > 0$.

$$L(\theta, \sigma^2|x) = \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}}\right)^n \exp\left[-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2\hat{\sigma}^2}\right]$$

Solution - Maximizing Numerator

Step 1, fix $\sigma^2$, likelihood is maximized when $\sum_{i=1}^n (x_i - \theta)^2$ is minimized over $\theta \leq \theta_0$.

$$\hat{\theta}_0 = \begin{cases} x & \text{if } x \leq \theta_0 \\ \theta_0 & \text{if } x > \theta_0 \end{cases}$$

Step 2: Now, we need to maximize likelihood (or log-likelihood) with respect to $\sigma^2$ and we substitute $\hat{\theta}_0$ for $\theta$.

$$l(\hat{\theta}, \sigma^2|x) = -\frac{n}{2} (\log 2\pi + \log \sigma^2) - \frac{\sum (x_i - \hat{\theta}_0)^2}{2\sigma^2}$$
$$\frac{\partial \log l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \hat{\theta}_0)^2}{2(\sigma^2)^2} = 0$$
$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^n (x_i - \hat{\theta}_0)^2}{n}$$

Combining the results together

$$\lambda(x) = \begin{cases} 1 & \text{if } \bar{x} \leq \theta_0 \\ \left(\frac{\hat{\sigma}_0^2}{\sigma_0^2}\right)^{n/2} & \text{if } \bar{x} > \theta_0 \end{cases}$$

Solution - Constructing LRT

LRT test rejects $H_0$ if and only if $\bar{x} > \theta_0$ and

$$\left(\frac{\hat{\sigma}_0^2}{\sigma_0^2}\right)^{n/2} \leq c$$
$$\left(\frac{\sum (x_i - \bar{x})^2}{n}\right)^{n/2} \leq c$$
$$\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \theta_0)^2} \leq c^*$$
$$\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \theta_0)^2 + n(\bar{x} - \theta_0)^2} \leq c^*$$
$$\frac{1}{1 + \frac{n(\bar{x} - \theta_0)^2}{\sum (x_i - \bar{x})^2}} \leq c^*$$
Solution - Constructing LRT (cont’d)

\[ n(x - \theta_0)^2 \geq c^{**} \]

\[ \frac{\bar{x} - \theta_0}{s_x/\sqrt{n}} \geq c^{***} \]

LRT test reject if \( \frac{\bar{x} - \theta_0}{s_x/\sqrt{n}} \geq c^{***} \)

The next step is specify \( c \) to get size \( \alpha \) test (omitted).

Unbiased Test

**Definition**

If a test always satisfies

\[ \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is false }) \geq \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is true }) \]

Then the test is said to be unbiased.

**Alternative Definition**

Recall that \( \beta(\theta) = \Pr(\text{reject } H_0) \). A test is unbiased if

\[ \beta(\theta') \geq \beta(\theta) \]

for every \( \theta' \in \Omega_1^c \) and \( \theta \in \Omega_0 \).

Example

\( X_1, \cdots, X_n \overset{i.i.d.}{\sim} N(\theta, \sigma^2) \) where \( \sigma^2 \) is known, testing \( H_0 : \theta \leq \theta_0 \) vs

\( H_1 : \theta > \theta_0 \).

LRT test rejects \( H_0 \) if \( \frac{\bar{x} - \theta_0}{s_x/\sqrt{n}} > c \).

\[ \beta(\theta) = \Pr \left( \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > c \right) \]

\[ = \Pr \left( \frac{\bar{X} - \theta + \theta - \theta_0}{\sigma/\sqrt{n}} > c \right) \]

\[ = \Pr \left( \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} + \frac{\theta - \theta_0}{\sigma/\sqrt{n}} > c \right) \]

\[ = \Pr \left( \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) \]

Note that \( X_i \sim N(\theta, \sigma^2), \bar{X} \sim N(\theta, \sigma^2/n) \), and \( \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \sim N(0, 1) \).

Example (cont’d)

Therefore, for \( Z \sim N(0, 1) \)

\[ \beta(\theta) = \Pr \left( Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) \]

Because the power function is increasing function of \( \theta \),

\[ \beta(\theta') \geq \beta(\theta) \]

always holds when \( \theta \leq \theta_0 < \theta' \). Therefore the LRTs are unbiased.
Summary

Today
- Examples of LRT
- LRT based on sufficient statistics
- LRT with nuisance parameters
- Unbiased Test

Next Lecture
- Uniformly Most Powerful Test