Biostatistics 602 - Statistical Inference
Lecture 15
Bayes Estimator

Hyun Min Kang

March 12th, 2013
Can Cramer-Rao bound be used to find the best unbiased estimator for any distribution?
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• When Cramer-Rao bound is attainable, can Cramer-Rao bound be used for find best unbiased estimator for any $\tau(\theta)$? If not, what is the restriction on $\tau(\theta)$?
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What is another way to find the best unbiased estimator?
Last Lecture

- Can Cramer-Rao bound be used to find the best unbiased estimator for any distribution? If not, in which cases?
- When Cramer-Rao bound is attainable, can Cramer-Rao bound be used for find best unbiased estimator for any \( \tau(\theta) \)? If not, what is the restriction on \( \tau(\theta) \)?
- What is another way to find the best unbiased estimator?
- Describe two strategies to obtain the best unbiased estimators for \( \tau(\theta) \), using complete sufficient statistics.
Recap - The power of complete sufficient statistics

Theorem 7.3.23

Let $T$ be a complete sufficient statistic for parameter $\theta$. Let $\phi(T)$ be any estimator based on $T$. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.
Finding UMUVE - Method 1

Use Cramer-Rao bound to find the best unbiased estimator for $\tau(\theta)$.

1. If "regularity conditions" are satisfied, then we have a Cramer-Rao bound for unbiased estimators of $\tau(\theta)$. 

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   - It helps to confirm an estimator is the best unbiased estimator of $\tau(\theta)$ if it happens to attain the CR-bound.
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   - It helps to confirm an estimator is the best unbiased estimator of $\tau(\theta)$ if it happens to attain the CR-bound.
   - If an unbiased estimator of $\tau(\theta)$ has variance greater than the CR-bound, it does NOT mean that it is not the best unbiased estimator.
Using Cramer-Rao bound to find the best unbiased estimator for $\tau(\theta)$.

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2. When "regularity conditions" are not satisfied, $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$ is no longer a valid lower bound.
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   - If an unbiased estimator of $\tau(\theta)$ has variance greater than the CR-bound, it does NOT mean that it is not the best unbiased estimator.

2. When "regularity conditions" are not satisfied, $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$ is no longer a valid lower bound.
   - There may be unbiased estimators of $\tau(\theta)$ that have variance smaller than $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$. 
Finding UMVUE - Method 2

Use complete sufficient statistic to find the best unbiased estimator for \( \tau(\theta) \).
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1. Find complete sufficient statistic $T$ for $\theta$. 

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2. Obtain $\phi(T)$, an unbiased estimator of $\tau(\theta)$ using either of the following two ways
   - Guess a function $\phi(T)$ such that $E[\phi(T)] = \tau(\theta)$. 
   - Guess an unbiased estimator $h(X)$ of $\tau(\theta)$. Construct $\phi(T) = E[h(X) | T]$, then $E[\phi(T)] = E[h(X)] = \tau(\theta)$. 
Use complete sufficient statistic to find the best unbiased estimator for \( \tau(\theta) \).

1. Find complete sufficient statistic \( T \) for \( \theta \).
2. Obtain \( \phi(T) \), an unbiased estimator of \( \tau(\theta) \) using either of the following two ways
   - Guess a function \( \phi(T) \) such that \( \mathbb{E}[\phi(T)] = \tau(\theta) \).
   - Guess an unbiased estimator \( h(X) \) of \( \tau(\theta) \). Construct \( \phi(T) = \mathbb{E}[h(X)|T] \), then \( \mathbb{E}[\phi(T)] = \mathbb{E}[h(X)] = \tau(\theta) \).
Frequentists vs. Bayesians

A biased view in favor of Bayesians at http://xkcd.com/1132/
Bayesian Statistic

Frequentist’s Framework

\[ \mathcal{P} = \{ \mathbf{X} \sim f_{\mathbf{X}}(\mathbf{x}|\theta), \theta \in \Omega \} \]
Frequentist’s Framework

\[ P = \{ X \sim f_X(x|\theta), \theta \in \Omega \} \]

Bayesian Statistic

- Parameter \( \theta \) is considered as a random quantity
Bayesian Statistic

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- Parameter \( \theta \) is considered as a random quantity
- Distribution of \( \theta \) can be described by probability distribution, referred to as \textit{prior} distribution
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Bayesian Statistic

- Parameter \( \theta \) is considered as a random quantity
- Distribution of \( \theta \) can be described by probability distribution, referred to as \textit{prior} distribution
- A sample is taken from a population indexed by \( \theta \), and the prior distribution is updated using information from the sample to get \textit{posterior} distribution of \( \theta \) given the sample.
Bayesian Framework

- Prior distribution of $\theta : \theta \sim \pi(\theta)$. 
Bayesian Framework

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- Sample distribution of $X$ given $\theta$.
  $$X|\theta \sim f(x|\theta)$$
Bayesian Framework

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- Joint distribution $X$ and $\theta$
  $$f(x, \theta) = \pi(\theta)f(x|\theta)$$
Bayesian Framework

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- Sample distribution of $\mathbf{X}$ given $\theta$.
  \[ \mathbf{X} | \theta \sim f(\mathbf{x} | \theta) \]

- Joint distribution $\mathbf{X}$ and $\theta$
  \[ f(\mathbf{x}, \theta) = \pi(\theta)f(\mathbf{x} | \theta) \]

- Marginal distribution of $\mathbf{X}$.
  \[ m(\mathbf{x}) = \int_{\theta \in \Omega} f(\mathbf{x}, \theta) \, d\theta = \int_{\theta \in \Omega} f(\mathbf{x} | \theta)\pi(\theta) \, d\theta \]
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- Posterior distribution of $\theta$ (conditional distribution of $\theta$ given $\mathbf{X}$)
  \[ \pi(\theta | \mathbf{x}) = \frac{f(\mathbf{x}, \theta)}{m(\mathbf{x})} = \frac{f(\mathbf{x} | \theta) \pi(\theta)}{m(\mathbf{x})} \quad \text{(Bayes’ Rule)} \]
Example

| Burglary ($\theta$) | $\Pr(\text{Alarm}|\text{Burglary}) = \Pr(X = 1|\theta)$ |
|---------------------|-------------------------------------------------|
| True ($\theta = 1$) | 0.95                                            |
| False ($\theta = 0$) | 0.01                                           |

Suppose that Burglary is an unobserved parameter ($\theta \in \{0, 1\}$), and Alarm is an observed outcome ($X = \{0, 1\}$).
Example

\[
\begin{array}{|c|c|}
\hline
\text{Burglary (}\theta\text{)} & \Pr(\text{Alarm|Burglary}) = \Pr(X = 1|\theta) \\
\hline
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\text{False (}\theta = 0\text{)} & 0.01 \\
\hline
\end{array}
\]

Suppose that Burglary is an unobserved parameter (\(\theta \in \{0, 1\}\)), and Alarm is an observed outcome (\(X = \{0, 1\}\)).

- Under Frequentist’s Framework,
  - If there was no burglary, there is 1% of chance of alarm ringing.
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  - One can come up with an estimator on $\theta$, such as MLE
Suppose that Burglary is an unobserved parameter ($\theta \in \{0, 1\}$), and Alarm is an observed outcome ($X = \{0, 1\}$).

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  - If there was a burglary, there is 95% of chance of alarm ringing.
  - One can come up with an estimator on $\theta$, such as MLE
  - However, given that alarm already rang, one cannot calculate the probability of burglary.
Suppose that we know that the chance of Burglary per household per night is $10^{-7}$.

$$\Pr(\theta = 1|X = 1) = \Pr(X = 1|\theta = 1) \frac{\Pr(\theta = 1)}{\Pr(X = 1)}$$  
(Bayes’ rule)
Inference Under Bayesian’s Framework

Leveraging Prior Information

Suppose that we know that the chance of Burglary per household per night is $10^{-7}$.

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$$= \Pr(X = 1|\theta = 1) \frac{\Pr(\theta = 1)}{\Pr(\theta = 1, X = 1) + \Pr(\theta = 0, X = 1)}$$
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So, even if the alarm rang, one can conclude that the burglary is unlikely to happen.
Inference Under Bayesian’s Framework

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\]

\[
= \frac{0.95 \times 10^{-7}}{0.95 \times 10^{-7} + 0.01 \times (1 - 10^{-7})} \approx 9.5 \times 10^{-6}
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So, even if alarm rang, one can conclude that the burglary is unlikely to happen.
What if the prior information is misleading?

Over-fitting to Prior Information

Suppose that, in fact, a thief found a security breach in my place and planning to break-in either tonight or tomorrow night for sure (with the same probability). Then the correct prior \( \Pr(\theta = 1) = 0.5 \).
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Suppose that, in fact, a thief found a security breach in my place and planning to break-in either tonight or tomorrow night for sure (with the same probability). Then the correct prior $\Pr(\theta = 1) = 0.5$.

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\Pr(\theta = 1|X = 1) = \frac{\Pr(X = 1|\theta = 1) \Pr(\theta = 1)}{\Pr(X = 1|\theta = 1) \Pr(\theta = 1) + \Pr(X = 1|\theta = 0) \Pr(\theta = 0)}
$$

$$
= \frac{0.95 \times 0.5}{0.95 \times 0.5 + 0.01 \times (1 - 0.5)} \approx 0.99
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**Over-fitting to Prior Information**

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= \frac{0.95 \times 0.5}{0.95 \times 0.5 + 0.01 \times (1 - 0.5)} \approx 0.99
\]

However, if we relied on the inference based on the incorrect prior, we may end up concluding that there are $> 99.9\%$ chance that this is a false alarm, and ignore it, resulting an exchange of one night of good sleep with quite a bit of fortune.
Advantages and Drawbacks of Bayesian Inference

Advantages over Frequentist’s Framework

- Allows making inference on the distribution of $\theta$ given data.
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Drawbacks of Bayesian Inference

- Misleading prior can result in misleading inference.
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Drawbacks of Bayesian Inference

- Misleading prior can result in misleading inference.
- Bayesian inference is often (but not always) prone to be "subjective”
- Bayesian inference could be sometimes unnecessarily complicated to interpret, compared to Frequentist’s inference.
Bayes Estimator

**Definition**

Bayes Estimator of $\theta$ is defined as the posterior mean of $\theta$. 

Example Problem:

Suppose we have $X_1, \ldots, X_n$ i.i.d. Bernoulli($p$) where $0 < p < 1$. Assume that the prior distribution of $p$ is Beta($\alpha, \beta$). Find the posterior distribution of $p$ and the Bayes estimator of $p$, assuming $\alpha$ and $\beta$ are known.
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$$E(\theta|x) = \int_{\theta \in \Omega} \theta \pi(\theta|x) d\theta$$
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$$E(\theta|\mathbf{x}) = \int_{\theta \in \Omega} \theta \pi(\theta|\mathbf{x}) d\theta$$

Example Problem

$X_1, \ldots, X_n \overset{i.i.d.}{\sim} \text{Bernoulli}(p)$ where $0 \leq p \leq 1$. Assume that the prior distribution of $p$ is $\text{Beta}(\alpha, \beta)$. Find the posterior distribution of $p$ and the Bayes estimator of $p$, assuming $\alpha$ and $\beta$ are known.
Solution (1/4)

Prior distribution of $p$ is

$$
\pi(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}
$$
Solution (1/4)

Prior distribution of $p$ is

$$\pi(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1 - p)^{\beta-1}$$

Sampling distribution of $X$ given $p$ is

$$f_X(x|p) = \prod_{i=1}^{n} \left\{ p^{x_i} (1 - p)^{1-x_i} \right\}$$
Solution (1/4)

Prior distribution of $p$ is

$$\pi(p) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}$$

Sampling distribution of $\mathbf{X}$ given $p$ is

$$f_{\mathbf{X}}(\mathbf{x}|p) = \prod_{i=1}^{n} \left\{ p^{x_i}(1 - p)^{1-x_i} \right\}$$

Joint distribution of $\mathbf{X}$ and $p$ is

$$f_{\mathbf{X}}(\mathbf{x}, p) = f_{\mathbf{X}}(\mathbf{x}|p)\pi(p)$$

$$= \prod_{i=1}^{n} \left\{ p^{x_i}(1 - p)^{1-x_i} \right\} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1 - p)^{\beta-1}$$
Solution (2/4)

The marginal distribution of \( X \) is

\[
m(X) = \int f(x, p) \, dp = \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\sum_{i=1}^n x_i + \alpha - 1} (1 - p)^{n - \sum_{i=1}^n x_i + \beta - 1} \, dp
\]
The marginal distribution of $X$ is

$$m(x) = \int f(x, p) \, dp = \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\sum_{i=1}^n x_i + \alpha - 1} (1 - p)^{n - \sum_{i=1}^n x_i + \beta - 1} dp$$

$$= \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\sum x_i + \alpha)\Gamma(n - \sum x_i + \beta)}{\Gamma(\alpha + \beta + n)} p^{\sum x_i + \alpha - 1} (1 - p)^{n - \sum x_i + \beta - 1} dp$$
The marginal distribution of $X$ is

$$m(x) = \int f(x, p) \, dp = \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{p^{\sum_{i=1}^n x_i + \alpha - 1}(1 - p)^{n - \sum_{i=1}^n x_i + \beta - 1}}{\sum_{i=1}^n x_i + \alpha + n - \sum_{i=1}^n x_i + \beta} \, dp$$

$$= \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\sum x_i + \alpha)\Gamma(n - \sum x_i + \beta)}{\Gamma(\sum x_i + \alpha + n - \sum x_i + \beta)} \frac{p^{\sum x_i + \alpha - 1}(1 - p)^{n - \sum x_i + \beta - 1}}{\Gamma(\sum x_i + \alpha + n - \sum x_i + \beta)} \, dp$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\sum_{i=1}^n x_i + \alpha)\Gamma(n - \sum_{i=1}^n x_i + \beta)}{\Gamma(\sum_{i=1}^n x_i + \alpha + n - \sum_{i=1}^n x_i + \beta)} \frac{p^{\sum x_i + \alpha - 1}(1 - p)^{n - \sum x_i + \beta - 1}}{\Gamma(\sum x_i + \alpha + n - \sum x_i + \beta)} \, dp$$

$$= \int_0^1 f_{\text{Beta}}(\sum x_i + \alpha, n - \sum x_i + \beta) (p) \, dp$$
Solution (2/4)

The marginal distribution of $X$ is

$$m(x) = \int f(x, p) dp = \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{p^{\sum_{i=1}^n x_i + \alpha - 1} (1 - p)^{n - \sum_{i=1}^n x_i + \beta - 1}}{\Gamma(\alpha + \beta + n)} dp$$

$$= \int_0^1 \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\sum x_i + \alpha) \Gamma(n - \sum x_i + \beta)}{\Gamma(\sum x_i + \alpha + n - \sum x_i + \beta)} \frac{p^{\sum_{i=1}^n x_i + \alpha - 1} (1 - p)^{n - \sum_{i=1}^n x_i + \beta - 1}}{\Gamma(\alpha + \beta + n)} dp$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\sum_{i=1}^n x_i + \alpha) \Gamma(n - \sum_{i=1}^n x_i + \beta)}{\Gamma(\alpha + \beta + n)} \int_0^1 f_{\text{Beta}}(\sum x_i + \alpha, n - \sum x_i + \beta) (p) dp$$

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$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha) \Gamma(\beta)} \frac{\Gamma(\sum_{i=1}^n x_i + \alpha) \Gamma(n - \sum_{i=1}^n x_i + \beta)}{\Gamma(\alpha + \beta + n)} \int_0^1 f_{\text{Beta}}(\sum x_i + \alpha, n - \sum x_i + \beta) (p) dp$$
Solution (3/4)

The posterior distribution of $\theta|x$:

$$
\pi(\theta|x) = \frac{f(x, p)}{m(x)}
$$
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$$
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$$

$$
= \frac{\Gamma(\alpha + \beta) \Gamma(\sum x_i + \alpha) \Gamma(n - \sum x_i + \beta)}{\Gamma(\alpha) \Gamma(\beta) \Gamma(\alpha + \beta + n)}
$$
The posterior distribution of $\theta|\mathbf{x}$:

$$
\pi(\theta|\mathbf{x}) = \frac{f(\mathbf{x}, p)}{m(\mathbf{x})} \left[ \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\sum x_i + \alpha - 1} (1 - p)^{n - \sum x_i + \beta - 1} \right]
= \frac{\Gamma(\alpha + \beta + n)}{\Gamma(\sum x_i + \alpha)\Gamma(n - \sum x_i + \beta)} p^{\sum x_i + \alpha - 1} (1 - p)^{n - \sum x_i + \beta - 1}
$$
Solution (4/4)

The Bayes estimator of $p$ is

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i + \alpha}{\sum_{i=1}^{n} x_i + \alpha + n - \sum_{i=1}^{n} x_i + \beta} = \frac{\sum_{i=1}^{n} x_i + \alpha}{\alpha + \beta + n}$$
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$$= \frac{\sum_{i=1}^{n} \frac{x_i}{n}}{\alpha + \beta + n} + \frac{\alpha}{\alpha + \beta} \frac{\alpha + \beta}{\alpha + \beta + n}$$
Solution (4/4)

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$$

$$
= \frac{\sum_{i=1}^{n} x_i}{n} \cdot \frac{n}{\alpha + \beta + n} + \frac{\alpha}{\alpha + \beta} \cdot \frac{\alpha + \beta}{\alpha + \beta + n}
$$

$$
= [\text{Guess about } p \text{ from data}] \cdot \text{weight}_1 + [\text{Guess about } p \text{ from prior}] \cdot \text{weight}_2
$$
Solution (4/4)

The Bayes estimator of $p$ is

$$
\hat{p} = \frac{\sum_{i=1}^{n} x_i + \alpha}{\sum_{i=1}^{n} x_i + \alpha + n - \sum_{i=1}^{n} x_i + \beta} = \frac{\sum_{i=1}^{n} x_i + \alpha}{\alpha + \beta + n}
$$

$$
= \frac{\sum_{i=1}^{n} x_i}{n} \frac{n}{\alpha + \beta + n} + \frac{\alpha}{\alpha + \beta} \frac{\alpha + \beta}{\alpha + \beta + n}
$$

$$
= [\text{Guess about } p \text{ from data}] \cdot \text{weight}_1 + [\text{Guess about } p \text{ from prior}] \cdot \text{weight}_2
$$

As $n$ increase, $\text{weight}_1 = \frac{n}{\alpha + \beta + n} = \frac{1}{\alpha + \beta + 1}$ becomes bigger and bigger and approaches to 1. In other words, influence of data is increasing, and the influence of prior knowledge is decreasing.
Is the Bayes estimator unbiased?

\[
E \left[ \frac{\sum_{i=1}^{n} x_i + \alpha}{\alpha + \beta + n} \right] = \frac{np + \alpha}{\alpha + \beta + n} \neq p
\]

Unless \( \frac{\alpha}{\alpha + \beta} = p \).
Is the Bayes estimator unbiased?

\[
E \left[ \frac{\sum_{i=1}^{n} \alpha}{\alpha + \beta + n} \right] = \frac{np + \alpha}{\alpha + \beta + n} \neq p
\]

Unless \( \frac{\alpha}{\alpha + \beta} = p \).

\[
\text{Bias} = \frac{np + \alpha}{\alpha + \beta + n} - p = \frac{\alpha - (\alpha + \beta)p}{\alpha + \beta + n}
\]

As \( n \) increases, the bias approaches to zero.
Sufficient statistic and posterior distribution

Posterior conditioning on sufficient statistics

If $T(X)$ is a sufficient statistic, then the posterior distribution of $\theta$ given $X$ is the same to the posterior distribution given $T(X)$.
Sufficient statistic and posterior distribution

Posterior conditioning on sufficient statistics

If $T(X)$ is a sufficient statistic, then the posterior distribution of $\theta$ given $X$ is the same to the posterior distribution given $T(X)$. In other words,

$$\pi(\theta|X) = \pi(\theta|T(X))$$
Conjugate family

Definition 7.2.15

Let \( \mathcal{F} \) denote the class of pdfs or pmfs for \( f(x|\theta) \). A class \( \Pi \) of prior distributions is a conjugate family of \( \mathcal{F} \), if the posterior distribution is the class \( \Pi \) for all \( f \in \mathcal{F} \), and all priors in \( \Pi \), and all \( x \in \mathcal{X} \).
Example: Beta-Binomial conjugate

Let

- \( X_1, \cdots, X_n | p \sim \text{Binomial}(m, p) \)
Example: Beta-Binomial conjugate

Let

- $X_1, \cdots, X_n | p \sim \text{Binomial}(m, p)$
- $\pi(p) \sim \text{Beta}(\alpha, \beta)$

where $m, \alpha, \beta$ is known.
Example: Beta-Binomial conjugate

Let

- $X_1, \cdots, X_n \mid p \sim \text{Binomial}(m, p)$
- $\pi(p) \sim \text{Beta}(\alpha, \beta)$

where $m, \alpha, \beta$ is known. The posterior distribution is

$$
\pi(p \mid x) \sim \text{Beta} \left( \sum_{i=1}^{n} x_i + \alpha, mn - \sum_{i=1}^{n} x_i + \beta \right)
$$
Example: Gamma-Poisson conjugate

- \( X_1, \cdots, X_n|\lambda \sim \text{Poisson}(\lambda) \)
Example: Gamma-Poisson conjugate

- $X_1, \cdots, X_n | \lambda \sim \text{Poisson}(\lambda)$
- $\pi(\lambda) \sim \text{Gamma}(\alpha, \beta)$
Example: Gamma-Poisson conjugate

- \( X_1, \ldots, X_n | \lambda \sim \text{Poisson}(\lambda) \)
- \( \pi(\lambda) \sim \text{Gamma}(\alpha, \beta) \)
- Prior:

\[
\pi(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}
\]
Example: Gamma-Poisson conjugate

- \( X_1, \ldots, X_n | \lambda \sim \text{Poisson}(\lambda) \)
- \( \pi(\lambda) \sim \text{Gamma}(\alpha, \beta) \)
- Prior:
  \[
  \pi(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}
  \]
- Sampling distribution
  \[
  f_X(x|\lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}
  \]
Gamma-Poisson conjugate (cont’d)

- Joint distribution of $X$ and $\lambda$.

\[
f(x|\lambda) \pi(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \frac{1}{\Gamma(\alpha) \beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta}
\]

\[
= e^{-n\lambda - \lambda/\beta} \lambda^{\sum x_i + \alpha - 1} \frac{1}{\prod_{i=1}^{n} x_i!} \frac{1}{\Gamma(\alpha) \beta^\alpha}
\]
Gamma-Poisson conjugate (cont’d)

- Joint distribution of $\mathbf{X}$ and $\lambda$.

\[
f(\mathbf{x}|\lambda)\pi(\lambda) = \left[ \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \right] \frac{1}{\Gamma(\alpha) \beta^{\alpha}} \lambda^{\alpha-1} e^{-\lambda/\beta} \\
= e^{-n\lambda-\lambda/\beta} \lambda^{\sum x_i + \alpha - 1} \frac{1}{\prod_{i=1}^{n} x_i!} \Gamma(\alpha) \beta^{\alpha}
\]

- Marginal distribution

\[
m(\mathbf{x}) = \int f(\mathbf{x}|\lambda)\pi(\lambda) d\lambda
\]
Posterior distribution (proportional to the joint distribution)

\[
\pi(\lambda|\mathbf{x}) = \frac{f(\mathbf{x}|\lambda)\pi(\lambda)}{m(\mathbf{x})} = e^{-n\lambda-\lambda/\beta} \lambda^{\sum x_i+\alpha-1} \frac{1}{\Gamma(\sum x_i + \alpha) \left(\frac{1}{n+\frac{1}{\beta}}\right)^{\sum x_i+\alpha}}
\]
Gamma-Poisson conjugate (cont’d)

- Posterior distribution (proportional to the joint distribution)

\[
\pi(\lambda | \mathbf{x}) = \frac{f(\mathbf{x} | \lambda) \pi(\lambda)}{m(\mathbf{x})}
\]

\[
= e^{-n\lambda - \lambda/\beta} \sum x_i + \alpha - 1 \frac{1}{\Gamma(\sum x_i + \alpha) \left( \frac{1}{n + \frac{1}{\beta}} \right)} \sum x_i + \alpha
\]

So, the posterior distribution is Gamma \( \left( \sum x_i + \alpha, \left( n + \frac{1}{\beta} \right)^{-1} \right) \).
Example: Normal Bayes Estimators

Let $X \sim \mathcal{N}(\theta, \sigma^2)$ and suppose that the prior distribution of $\theta$ is $\mathcal{N}(\mu, \tau^2)$. Assuming that $\sigma^2, \mu^2, \tau^2$ are all known, the posterior distribution of $\theta$ also becomes normal, with mean and variance given by
Example: Normal Bayes Estimators

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$$E[\theta | x] = \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu$$
Example: Normal Bayes Estimators

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$$
E[\theta|x] = \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu
$$

$$
\text{Var}(\theta|x) = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}
$$
Example: Normal Bayes Estimators

Let \( X \sim \mathcal{N}(\theta, \sigma^2) \) and suppose that the prior distribution of \( \theta \) is \( \mathcal{N}(\mu, \tau^2) \). Assuming that \( \sigma^2, \mu^2, \tau^2 \) are all known, the posterior distribution of \( \theta \) also becomes normal, with mean and variance given by

\[
\begin{align*}
\mathbb{E}[\theta|x] &= \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu \\
\text{Var}(\theta|x) &= \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}
\end{align*}
\]

- The normal family is its own conjugate family.
Example: Normal Bayes Estimators

Let $X \sim \mathcal{N} (\theta, \sigma^2)$ and suppose that the prior distribution of $\theta$ is $\mathcal{N} (\mu, \tau^2)$. Assuming that $\sigma^2, \mu^2, \tau^2$ are all known, the posterior distribution of $\theta$ also becomes normal, with mean and variance given by

$$E[\theta | x] = \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu$$

$$\text{Var}(\theta | x) = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}$$

- The normal family is its own conjugate family.
- The Bayes estimator for $\theta$ is a linear combination of the prior and sample means.
Example: Normal Bayes Estimators

Let $X \sim \mathcal{N}(\theta, \sigma^2)$ and suppose that the prior distribution of $\theta$ is $\mathcal{N}(\mu, \tau^2)$. Assuming that $\sigma^2, \mu^2, \tau^2$ are all known, the posterior distribution of $\theta$ also becomes normal, with mean and variance given by

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E[\theta|x] = \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu
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\[
\text{Var}(\theta|x) = \frac{\sigma^2 \tau^2}{\sigma^2 + \tau^2}
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- The normal family is its own conjugate family.
- The Bayes estimator for $\theta$ is a linear combination of the prior and sample means.
- As the prior variance $\tau^2$ approaches to infinity, the Bayes estimator tends toward to sample mean.
Example: Normal Bayes Estimators

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$$E[\theta|x] = \frac{\tau^2}{\tau^2 + \sigma^2} x + \frac{\sigma^2}{\sigma^2 + \tau^2} \mu$$

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- The normal family is its own conjugate family.
- The Bayes estimator for $\theta$ is a linear combination of the prior and sample means.
- As the prior variance $\tau^2$ approaches to infinity, the Bayes estimator tends toward to sample mean
  - As the prior information becomes more vague, the Bayes estimator tends to give more weight to the sample information.
Summary

Today

- Bayesian Statistics
- Bayes Estimator
- Conjugate family
Summary

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- Bayesian Statistics
- Bayes Estimator
- Conjugate family

Next Lecture

- Bayesian Risk Functions
- Consistency