Biostatistics 615/815 Lecture 5: Divide and Conquer Algorithms
Sorting Algorithms

Hyun Min Kang

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Recap - An example C++ class

```cpp
#include <iostream>
#include <cmath>

class Point {
    public:
        double x;
        double y;
        // A constructor can call constructor of each member variable
        Point(double px, double py) : px(x), py(y) {}
        // equivalent to -- Point(double px, double py) : x(px), y(py) {}
        double distanceFromOrigin() { return sqrt(x*x + y*y); }
};

int main(int argc, char** argv) {
    Point p(3,4); // calls constructor with two arguments
    std::cout << p.distanceFromOrigin() << std::endl; // prints 5
}
```
Recap: STL in practice

```cpp
#include <iostream>
#include <string>
#include <vector>

int main(int argc, char** argv) {
    std::vector<std::string> vArgs; // vector of strings
    for(int i=1; i < argc; ++i) {
        vArgs.push_back(argv[i]); // append each arguments to the vector
    }
    std::sort(vArgs.begin(),vArgs.end()); // sort the vector in alphanumeric order
    std::cout << "Sorted arguments :"; // print the sorted arguments
    for(int i=0; i < vArgs.size(); ++i) { std::cout << " " << vArgs[i]; }
    std::cout << std::endl;
    return 0;
}
```

A running example

```
user@host:~/> ./sortedEcho Hello, World! hello, world! 2 3 5 60 1
Sorted arguments : 1 2 3 5 60 Hello, World! hello, world!
```
Recap: Euclid’s algorithm

Algorithm $\text{GCD}$

**Data**: Two integers $a$ and $b$

**Result**: The greatest common divisor (GCD) between $a$ and $b$

if $a$ divides $b$ then
  return $a$
else
  Find the largest integer $t$ such that $at + r = b$;
  return $\text{GCD}(r, a)$
end
Recap: Euclid’s algorithm

Algorithm $GCD$

**Data:** Two integers $a$ and $b$

**Result:** The greatest common divisor (GCD) between $a$ and $b$

```python
if a divides b then
  return a
else
  Find the largest integer $t$ such that $at + r = b$;
  return $GCD(r,a)$
end
```

Function $gcd()$

```python
int gcd (int a, int b) {
  if (a == 0) return b; // equivalent to returning a when b % a == 0
  else return gcd(b % a, a);
}
```
Divide-and-conquer algorithms

Solve a problem recursively, applying three steps at each level of recursion

**Divide**  the problem into a number of subproblems that are smaller instances of the same problem

**Conquer**  the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.

**Combine**  the solutions to subproblems into the solution for the original problem
// assuming a is sorted, return index of array containing the key, // among a[start...end]. Return -1 if no key is found
int binarySearch(std::vector<int>& a, int key, int start, int end) {
    if ( start > end ) return -1; // search failed
    int mid = (start+end)/2;
    if ( key == a[mid] ) return mid; // terminate if match is found
    if ( key < a[mid] ) // divide the remaining problem into half
        return binarySearch(a, key, start, mid-1);
    else
        return binarySearch(a, key, mid+1, end);
}
// find maximum within an a[start..end]
int findMax(std::vector<int>& a, int start, int end) {
    if ( start == end ) return a[start];  // conquer small problem directly
    else {
        int mid = (start+end)/2;
        int leftMax = findMax(a,start,mid);  // divide the problem into half
        int rightMax = findMax(a,mid+1,end);
        return ( leftMax > rightMax ? leftMax : rightMax );  // combine solutions
    }
}
# include <iostream>
# include <fstream>
# include <vector>

int main(int argc, char** argv) { // sorting software using std::sort
  int tok;
  std::vector<int> v;
  if ( argc > 1 ) { // if argument is given, read from file
    std::ifstream fin(argv[1]);
    while( fin >> tok ) { v.push_back(tok); }  
    fin.close();
  }
  else { // read from standard input if no argument is specified
    while( std::cin >> tok ) { v.push_back(tok); }  
  }
  std::sort(v.begin(), v.end()); // Sort using the algorithm in STL
  for(int i=0; i < v.size(); ++i) {
    std::cout << v[i] << std::endl; // print out the content
  }
  return 0;
}
Reading from Files: insertionSort.cpp

```cpp
#include <iostream>
#include <fstream>
#include <vector>

void insertionSort(std::vector<int> &v); // insertionSort as defined before

int main(int argc, char** argv) {
    int tok;
    std::vector<int> v;
    if ( argc > 1 ) {
        std::ifstream fin(argv[1]);
        while ( fin >> tok ) { v.push_back(tok); }
        fin.close();
    } else {
        while ( std::cin >> tok ) { v.push_back(tok); }
    }
    insertionSort(v); // differs from stdSort in only this part
    for(int i=0; i < v.size(); ++i) {
        std::cout << v[i] << std::endl;
    }
    return 0;
}
```

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STL Use in **INSERTIONSORT** Algorithm

### insertionSort.cpp - insertionSort() function

```cpp
// perform insertion sort on A
void insertionSort(std::vector<int>& A) { // call-by-reference
    for(int j=1; j < A.size(); ++j) { // 0-based index
        int key = A[j]; // key element to relocate
        int i = j-1; // index to be relocated
        while( (i >= 0) && (A[i] > key) ) { // find position to relocate
            A[i+1] = A[i]; // shift elements
            --i; // update index to be relocated
        }
        A[i+1] = key; // relocate the key element
    }
}
```
Running time comparison

Running example with 100,000 elements (in UNIX or MacOS)

```
user@host:~/> time cat src/sample.input.txt | src/stdSort > /dev/null
real 0m0.430s
user 0m0.281s
sys 0m0.130s

user@host:~/> time cat src/sample.input.txt | src/insertionSort > /dev/null
real 1m8.795s
user 1m8.181s
sys 0m0.206s
```
### Divide and conquer algorithm

**Divide**  
Divide the $n$ element sequence to be sorted into two subsequences of $n/2$ elements each

**Conquer**  
Sort the two subsequences recursively using merge sort

**Combine**  
Merge the two sorted subsequences to produce the sorted answer
mergeSort.cpp - main()

```cpp
#include <iostream>
#include <vector>
#include <climits>

void mergeSort(std::vector<int>& a, int p, int r); // defined later
void printArray(std::vector<int>& A); // same as insertionSort
// same to insertionSort.cpp except for one line

int main(int argc, char** argv) {
    std::vector<int> v;
    int tok;
    while ( std::cin >> tok ) {
        v.push_back(tok);
    }
    std::cout << "Before sorting: ";
    printArray(v);
    mergeSort(v, 0, v.size()-1); // differs from insertionSort.cpp
    std::cout << "After sorting: ";
    printArray(v);
    return 0;
}
```

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September 20th, 2011  
13 / 30
mergeSort.cpp - merge() function

// merge piecewise sorted a[p..q] a[q+1..r] into a sorted a[p..r]
void merge(std::vector<int>& a, int p, int q, int r) {
    std::vector<int> aL, aR; // copy a[p..q] to aL and a[q+1..r] to aR
    for(int i=p; i <= q; ++i) aL.push_back(a[i]);
    for(int i=q+1; i <= r; ++i) aR.push_back(a[i]);
aL.push_back(INT_MAX); // append additional value to avoid out-of-bound
    aR.push_back(INT_MAX);
    // pick smaller one first from aL and aR and copy to a[p..r]
    for(int k=p, i=0, j=0; k <= r; ++k) {
        if ( aL[i] <= aR[j] ) {
            a[k] = aL[i];
            ++i;
        } else {
            a[k] = aR[j];
            ++j;
        }
    }
}
mergeSort.cpp - mergeSort() function

```cpp
void mergeSort(std::vector<int>& a, int p, int r) {
    if ( p < r ) {
        int q = (p+r)/2;  // find a point to divide the problem
        mergeSort(a, p, q);  // divide-and-conquer
        mergeSort(a, q+1, r);  // divide-and-conquer
        merge(a, p, q, r);  // combine the solutions
    }
}
```
Time Complexity of Merge Sort

If $n = 2^m$

$$T(n) = \begin{cases} 
  c & \text{if } n = 1 \\
  2T(n/2) + cn & \text{if } n > 1 
\end{cases}$$

$$T(n) = \sum_{i=1}^{m} cn = cmn = cn \log_2(n) = \Theta(n \log_2 n)$$
Time Complexity of Merge Sort

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\[
T(n) = \sum_{i=1}^{m} cn = cmn = cn \log_2(n) = \Theta(n \log_2 n)
\]

For arbitrary \( n \)

\[
T(n) = \begin{cases} 
  c & \text{if } n = 1 \\
  T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + cn & \text{if } n > 1 
\end{cases}
\]

\[
 cn \lfloor \log_2 n \rfloor \leq T(n) \leq cn \lceil \log_2 n \rceil 
\]

\[
 T(n) = \Theta(n \log_2 n)
\]
Running time comparison

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user@host:~/> time cat src/sample.input.txt | src/insertionSort > /dev/null
real 1m8.795s
user 1m8.181s
sys 0m0.206s

user@host:~/> time cat src/sample.input.txt | src/mergeSort > /dev/null
real 0m0.898s
user 0m0.755s
sys 0m0.131s
```
Quicksort Overview

- Worst-case time complexity is $\Theta(n^2)$
- Expected running time is $\Theta(n \log_2 n)$.
- But in practice mostly performs the best
QuickSort

QuickSort Overview

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- But in practice mostly performs the best

Divide and conquer algorithm

**Divide**  Parition (rearrange) the array $A[p..r]$ into two subarrays

- Each element of $A[p..q-1] \leq A[q]$
- Each element of $A[q+1..r] \geq A[q]$

Compute the index $q$ as part of this partitioning procedure

**Conquer**  Sort the two subarrays by recursively calling quicksort

**Combine**  Because the subarrays are already sorted, no work is needed to combine them. The entire array $A[p..r]$ is now sorted
QuickSort Algorithm

Algorithm QUICKSORT

Data: array $A$ and indices $p$ and $r$

Result: $A[p..r]$ is sorted

if $p < r$ then
    $q = \text{PARTITION}(A, p, r)$;
    QUICKSORT($A, p, q - 1$);
    QUICKSORT($A, q + 1, r$);
end
QuickSort Algorithm

Algorithm PARTITION

Data: array \( A \) and indices \( p \) and \( r \)

Result: Returns \( q \) such that \( A[p..q-1] \leq A[q] \leq A[q+1..r] \)

\[ x = A[r]; \]
\[ i = p - 1; \]

\textbf{for} \( j = p \) \textbf{to} \( r - 1 \) \textbf{do}

\hspace{1em} \textbf{if} \( A[j] \leq x \) \textbf{then}

\hspace{2em} \( i = i + 1; \)

\hspace{2em} \text{EXCHANGE}(A[i], A[j]);

\hspace{1em} \textbf{end}

\textbf{end}

\text{EXCHANGE}(A[i + 1], A[r]);

\textbf{return} \( i + 1; \)
How **PARTITION** Algorithm Works

![Diagram of the PARTITION algorithm](image-url)

- **PARTITION** Algorithm Works by dividing the array into two parts: elements less than or equal to a pivot and elements greater than the pivot.
- The algorithm proceeds by recursively applying the same process to each partition until the entire array is sorted.

---

**Partition Algorithm**

1. Choose a pivot element from the array.
2. Initialize two pointers, `i` and `j`, to the start and end of the array.
3. Move `i` to the right until an element greater than the pivot is found.
4. Move `j` to the left until an element less than or equal to the pivot is found.
5. Swap the elements at `i` and `j`.
6. Repeat steps 3-5 until `i` and `j` cross.
7. Place the pivot element in its correct position.
8. Recursively apply the partition algorithm to the subarrays on the left and right of the pivot.
Implementation of **QUICKSORT** Algorithm

```cpp
// quickSort function
// The main function is the same to mergeSort.cpp except for the function name
void quickSort(std::vector<int>& A, int p, int r) {
  if ( p < r ) { // immediately terminate if subarray size is 1
    int piv = A[r]; // take a pivot value
    int i = p-1; // p-i-1 is the # elements < piv among A[p..j]
    int tmp;
    for(int j=p; j < r; ++j) {
      if ( A[j] < piv ) { // if smaller value is found, increase q (=i+1)
        ++i;
      }
    }
    quickSort(A, p, i);
    quickSort(A, i+2, r);
  }
}
```
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user@host:~/> time cat src/sample.input.txt | src/mergeSort > /dev/null
real 0m0.898s
user 0m0.755s
sys 0m0.131s

user@host:~/> time cat src/sample.input.txt | src/quickSort > /dev/null
real 0m0.427s
user 0m0.285s
sys 0m0.129s
```
Lower bounds for comparison sorting

CLRS Theorem 8.1

Any comparison-based sort algorithm requires $\Omega(n \log n)$ comparisons in the worst case.
Lower bounds for comparison sorting

CLRS Theorem 8.1
Any comparison-based sort algorithm requires $\Omega(n \log n)$ comparisons in the worst case

An informal proof
- Any comparison sort algorithm can be represented as a binary decision tree, where each node represents a comparison. Each path from the root to leaf represents possible series of comparisons to sort a sequence.
Lower bounds for comparison sorting

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An informal proof

- Any comparison sort algorithm can be represented as a binary decision tree, where each node represents a comparison. Each path from the root to leaf represents possible series of comparisons to sort a sequence.
- Each leaf of the decision tree represents one of $n!$ possible permutations of input sequences.
Lower bounds for comparison sorting

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- Any comparison sort algorithm can be represented as a binary decision tree, where each node represents a comparison. Each path from the root to leaf represents possible series of comparisons to sort a sequence.
- Each leaf of the decision tree represents one of $n!$ possible permutations of input sequences.
- We have $n! \leq l \leq 2^h$, where $l$ is the number of leaf nodes, and $h$ is the height of the tree, equivalent to the # of comparisons.
Lower bounds for comparison sorting

CLRS Theorem 8.1

Any comparison-based sort algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

An informal proof

- Any comparison sort algorithm can be represented as a binary decision tree, where each node represents a comparison. Each path from the root to leaf represents possible series of comparisons to sort a sequence.
- Each leaf of the decision tree represents one of $n!$ possible permutations of input sequences.
- We have $n! \leq l \leq 2^h$, where $l$ is the number of leaf nodes, and $h$ is the height of the tree, equivalent to the $\#$ of comparisons.
- Then it implies $h \geq \log(n!) = \Theta(n \log n)$. 
Example decision-tree representing InsertionSort
Finding faster sorting methods

Sorting faster than $\Theta(n \log n)$

- Comparison-based sorting algorithms cannot be faster than $\Theta(n \log n)$
- Sorting algorithms NOT based on comparisons may be faster
Finding faster sorting methods

### Sorting faster than $\Theta(n \log n)$

- Comparison-based sorting algorithms cannot be faster than $\Theta(n \log n)$
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### Linear time sorting algorithms

- Counting sort
- Radix sort
- Bucket sort
A linear sorting algorithm: Counting sort

A restrictive input setting

- The input sequences have a finite range with many expected duplication.
- For example, each element of input sequences is one digit number, and your input sequences are millions.

Key idea

1. Scan through each input sequence and count number of occurrences of each possible input value.
2. From the smallest to largest possible input value, output each value repeatedly by its stored count.
Another linear sorting algorithm: Radix sort

Key idea
- Sort the input sequence from the last digit to the first repeatedly using a linear sorting algorithm such as CountingSort.
- Applicable to integers within a finite range.
## Sorting Algorithms

- **Insertion Sort**: $\Theta(n^2)$, loop invariant property
- **Merge Sort**: $\Theta(n \log n)$, straightforward divide and conquer, a little memory overhead
- **Quicksort**: $\Theta(n^2)$ worst case, $\Theta(n \log n)$ expected, partition algorithm, practically fast
- **Count Sort**: Linear algorithm, may require much memory
- **Radix Sort**: $\Theta(nk)$ with $k$ digits
Next Lecture

Sorting Algorithms
- Radix Sort

Overview of elementary data structures
- Array
- Sorted array
- Linked list
- Binary search tree
- Hash table