Biostatistics 602 - Statistical Inference
Lecture 05
Complete Statistics

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1. What is an ancillary statistic for θ?
2. Can an ancillary statistic be a sufficient statistic?
3. What are the location parameter and the scale parameter?
4. In which case ancillary statistics would be helpful?

Last Lecture : Ancillary Statistics

Definition 6.2.16

A statistic $S(\mathbf{X})$ is an *ancillary statistic* if its distribution does not depend on θ.

Examples of Ancillary Statistics

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where σ^2 is known.
- $s_{\mathbf{X}}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_1 - \overline{X})^2$ is an ancillary statistic
- $X_1 - X_2 \sim \mathcal{N}(0, 2\sigma^2)$ is ancillary.
- $(X_1 + X_2)/2 - X_3 \sim \mathcal{N}(0, 1.5\sigma^2)$ is ancillary.
- $\frac{(n-1)s_{\mathbf{X}}^2}{\sigma^2} \sim \chi_{n-1}^2$ is ancillary.

Example : Uniform Ancillary Statistics

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1)$.
- Show that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic.

Possible Strategies

- Method 1 : Obtain the distribution of R and show that it is independent of θ.
- Method 2 : Represent R as a function of ancillary statistics, which is independent of θ.

Method 2 - A Simpler Proof

$$f_X(x|\theta) \;=\; I(\theta < x < \theta + 1) = I(0 < x - \theta < 1)$$

Let $Y_i = X_i - \theta \sim \text{Uniform}(0,1)$. Then $X_i = Y_i + \theta$, $|\frac{dx}{dy}| = 1$ holds.

$$f_Y(y) \;=\; I(0 < y + \theta - \theta < 1)|\tfrac{dx}{dy}| = I(0 < y < 1)$$

Then, the range statistic R can be rewritten as follows.

$$R \;=\; X_{(n)} - X_{(1)} = (Y_{(n)} + \theta) - (Y_{(1)} + \theta) = Y_{(n)} - Y_{(1)}$$

As $Y_{(n)} - Y_{(1)}$ is a function of $Y_1, \cdots, Y_n$. Any joint distribution of $Y_1, \cdots, Y_n$ does not depend on θ. Therefore, R is an ancillary statistic.

Location-Scale Family and Parameters

Definition 3.5.5

Let $f(x)$ be any pdf. Then for any μ, $-\infty < \mu < \infty$, and any $\sigma > 0$ the family of pdfs $f((x - \mu)/\sigma)/\sigma$, indexed by the parameter (μ, σ) is called the *location-scale family with standard pdf $f(x)$*, and μ is called the *location parameter* and σ is called the *scale parameter* for the family.

Example

- $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sim \mathcal{N}(0, 1)$
- $f((x - \mu)/\sigma)/\sigma = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \sim \mathcal{N}(\mu, \sigma^2)$

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
  - In other words, $g(T) = 0$ almost surely.
- Equivalently, $T(\mathbf{X})$ is called a *complete statistic*

Example

- $T(X) \sim \mathcal{N}(0, 1)$
- $g_1(T(\mathbf{X})) = 0 \implies \Pr[g_1(T(\mathbf{X})) = 0] = 1$.
- $g_2(T(\mathbf{X})) = I(T(\mathbf{X}) = 0) \implies \Pr[g_2(T(\mathbf{X})) = 0] = 1 - \Pr[T(\mathbf{X}) = 0]$. In this case, $g_2(T(\mathbf{X})) = 0$ is almost surely true.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.
- For example, $X \sim \mathcal{N}(0, 1)$ and $g(x) = x$ makes $E[g(\mathbf{X})] = E\mathbf{X} = 0$, but $\Pr(g(\mathbf{X}) = 0) = 0$ instead of 1.
- The above example is only for a particular distribution, not a family of distributions.
- If $X \sim \mathcal{N}(\theta, 1), -\infty < \theta < \infty$, then no function of X except for $g(\mathbf{X}) = 0$ satisfies $E[g(\mathbf{X})|\theta]$ for all θ.
- Therefore, the family of $\mathcal{N}(\theta, 1)$ distributions, $-\infty < \theta < \infty$, is complete.

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.

- Requiring $g(T)$ to satisfy the definition puts a restriction on g. The larger the family of pdfs/pmfs, the greater the restriction on g. When the family of pdfs/pmfs is augmented to the point that $E[g(T)] = 0$ for all θ, it rules out all g except for the trivial $g(T) = 0$, then the family is said to be complete.

Example - Poisson distribution

Problem

- Suppose $\mathcal{T} = \left\{ f_T : f_T(t|\lambda) = \frac{\lambda^t e^{-\lambda}}{t!} \right\}$ for $t \in \{0, 1, 2, \cdots\}$. Let $\lambda \in \Omega = \{1, 2\}$. Show that this family is NOT complete

Proof strategy

- We need to find a counter example,

- which is a function g such that $E[g(T)|\lambda] = 0$ for $\lambda = 1, 2$ but $g(T) \neq 0$.

Poisson distribution example : Proof

The function g must satisfy

$$E[g(T)|\lambda] = \sum_{t=0}^{\infty} g(t) \frac{\lambda^t e^{-\lambda}}{t!} = 0$$

for $\lambda \in \{1, 2\}$. Thus,

$$\begin{cases} E[g(T)|\lambda = 1] &= \sum_{t=0}^{\infty} g(t) \frac{1^t e^{-1}}{t!} = 0 \\ \\ E[g(T)|\lambda = 2] &= \sum_{t=0}^{\infty} g(t) \frac{2^t e^{-2}}{t!} = 0 \end{cases}$$

The above equation can be rewritten as

$$\begin{cases} \sum_{t=0}^{\infty} g(t)/t! &= 0 \\ \\ \sum_{t=0}^{\infty} 2^t g(t)/t! &= 0 \end{cases}$$

Poisson distribution example : Proof (cont'd)

Define $g(t)$ as

$$g(t) = \begin{cases} 2 & t = 0 \vee t = 2 \\ -3 & t = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\sum_{t=0}^{\infty} g(t)/t! = g(0)/0! + g(1)/1! + g(2)/2! = 2 - 3 + 2/2 = 0$$

$$\sum_{t=0}^{\infty} 2^t g(t)/t! = g(0)/0! + 2g(1)/1! + 2^2 g(2)/2! = 2 - 6 + 8/2 = 0$$

There exists a non-zero function g that satisfies $E[g(T)\lambda] = 0$ for all $\lambda \in \Omega$. Therefore this family is NOT complete.

Another example with Poisson distribution

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda), \lambda > 0$.
- Show that $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete statistic.

Proof strategy

- Need to find the distribution of $T(\mathbf{X})$
- Show that there is no non-zero function g such that $E[g(T)|\lambda] = 0$ for all λ.

Proof : Finding the moment-generating function of X

$$
\begin{aligned}
M_X(s) &= E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda} \lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^s \lambda)^x}{x!} e^{-e^s \lambda} e^{e^s \lambda} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} e^{e^s \lambda} \frac{(e^s \lambda)^x e^{-e^s \lambda}}{x!} \\
&= e^{\lambda} e^{e^s \lambda} \sum_{x=0}^{\infty} f_{Poisson}(x|e^s \lambda) \\
&= e^{\lambda(e^s - 1)}
\end{aligned}
$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$
\begin{aligned}
M_T(s) &= E(e^{s \sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i} \right) \\
&= \prod_{i=1}^n E\left(e^{sX_i} \right) = \left[E\left(e^{sX_i} \right) \right]^n \\
&= \left[e^{-\lambda(e^s - 1)} \right]^n = e^{n\lambda(e^s - 1)}
\end{aligned}
$$

Theorem 2.3.11 (b)

Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exists. If the moment generating functions exists and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

By Theorem 2.3.11, $T(\mathbf{X}) \sim \text{Poisson}(n\lambda)$.

Proof : Showing $E[g(T)|\lambda] = 0 \iff \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$
\begin{aligned}
E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\
&= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0
\end{aligned}
$$

Which is equivalent to

$$
\sum_{t=0}^{\infty} \frac{g(t)n^t}{t!} \lambda^t = 0
$$

for all $\lambda > 0$. Because the function above is a power series expansion of λ, $g(t)n^t/t! = 0$ for all t. and $g(t) = 0$ for all t. Therefore $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete statistic.

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$, $\theta > 0$, $\Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is $F_X(x) = \frac{x}{\theta} I(0 < x < \theta) + I(x \geq \theta)$.

$$
\begin{aligned}
f_T(t|\theta) &= \frac{n!}{(n-1)!} f_X(t) F_X(t)^{n-1} \\
&= \frac{n}{\theta} \left(\frac{t}{\theta}\right)^{n-1} I(0 < t < \theta) = n\theta^{-n} t^{n-1} I(0 < t < \theta)
\end{aligned}
$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$
\begin{aligned}
E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\
&= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0
\end{aligned}
$$

Taking derivative of both sides,

$$
\frac{n}{\theta^n} g(\theta)\theta^{n-1} - \frac{n^2}{\theta^{n+1}} \int_0^\theta g(t) t^{n-1} dt = 0
$$

$$
\frac{ng(\theta)}{\theta} = \frac{n}{\theta} \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = \frac{n}{\theta} E[g(T)|\theta] = 0
$$

Because $g(T) = 0$ holds for all $\theta > 0$, $T(\mathbf{X}) = X_{(n)}$ is a complete statistic.

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$
\begin{aligned}
E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\
&= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0
\end{aligned}
$$

$$
\int_0^\theta g(t) t^{n-1} dt = 0
$$

Taking derivative of both sides,

$$
\begin{aligned}
g(\theta)\theta^{n-1} &= 0 \\
g(\theta) &= 0
\end{aligned}
$$

for all $\theta > 0$. Because $g(T) = 0$ holds for all $\theta > 0$, $T(\mathbf{X}) = X_{(n)}$ is a complete statistic.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1)$, $\theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.
- Show that $\mathbf{T}(\mathbf{X})$ is not a complete statistic.

Proof - Using a range statistic

Define $R = X_{(n)} - X_{(1)}$. We have previously shown that

$$
f_R(r|\theta) = n(n-1) r^{(n-2)} (1 - r) \qquad , 0 < r < 1
$$

Then $R \sim \text{Beta}(n-1, 2)$, and $E[R|\theta] = \frac{n-1}{n+1}$.

Proof

Define $g(\mathbf{T}(X)) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$

$$
\begin{aligned}
E[g(\mathbf{T})|\theta] &= E[X_{(n)} - X_{(1)}|\theta] - \frac{n-1}{n+1} \\
&= \frac{n-1}{n+1} - \frac{n-1}{n+1} = 0
\end{aligned}
$$

Therefore, there exist a $g(\mathbf{T})$ such that $\Pr[g(\mathbf{T})|\theta] < 1$ for all θ, so $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not a complete statistic.

Even Simpler Proof

- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.
- Define $g(\mathbf{T}) = X_{(n)} - X_{(1)} - E(R)$. Note that $E(R)$ is constant to θ.
- Then $E[g(\mathbf{T})|\theta] = E(R) - E(R) = 0$, so T is not a complete statistic.

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$. Is $T(X) = X$ a complete statistic?

Solution

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta \in \mathbb{N}$.
Note that $f_X(x) = \frac{1}{\theta}I(x \in \{1, \cdots, \theta\}) = \frac{1}{\theta}I_{\mathbb{N}_\theta}(x)$.

$$
\begin{aligned}
E[g(T)|\theta] &= E[g(X)|\theta] = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0 \\
\sum_{x=1}^{\theta} g(x) &= 0
\end{aligned}
$$

Solution (cont'd)

for all $\theta \in \mathbb{N}$, which implies

- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$
- if $\theta = 2$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = g(2) = 0$.
- $\vdots$
- if $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + \cdots + g(k-1) + g(2) = g(k) = 0$.

Therefore, $g(x) = 0$ for all $x \in \mathbb{N}$, and $T(X) = X$ is a complete statistic for $\theta \in \Omega = \mathbb{N}$.

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where
$\theta \in \Omega = \mathbb{N} - \{n\}$. Is $T(X) = X$ a complete statistic?

Solution

Define a nonzero $g(x)$ as follows

$$g(x) = \begin{cases} 1 & x = n \\ -1 & x = n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[g(T)|\theta] = \frac{1}{\theta}\sum_{x=1}^{\theta} g(x) = \begin{cases} 0 & \theta \neq n \\ \frac{1}{\theta} & \theta = n \end{cases}$$

Because Ω does not include n, $g(x) = 0$ for all $\theta \in \Omega = \mathbb{N} - \{n\}$, and
$T(X) = X$ is not a complete statistic.

Summary

Today - Complete Statistics

- Examples of complete statistics
- Two Poisson distribution examples
- Two Uniform distribution examples
- Example of barely complete statistics.

Next Lecture

- Basu's Theorem