Biostatistics 602 - Statistical Inference
Lecture 14
Obtaining Best Unbiased Estimator

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Last Lecture

- For single-parameter exponential family, is Cramer-Rao bound always attainable?
- How about exponential family with two or more parameters?
- For any statistic $T(\mathbf{X})$, does $\phi(T)$ always result in a better unbiased estimator than W? Why?
- What is the Rao-Blackwell Theorem?
- Is the best unbiased estimator (UMVUE) for $\tau(\theta)$ unique?
- What is the relationship between the UMVUE and the unbiased estimators of zero?

Rao-Blackwell Theorem

Theorem 7.3.17

Let $W(\mathbf{X})$ be any unbiased estimator of $\tau(\theta)$, and T be a sufficient statistic for θ. Define $\phi(T) = E[W|T]$. Then the followings hold.

1. $E[\phi(T)|\theta] = \tau(\theta)$
2. $\text{Var}[\phi(T)|\theta] \leq \text{Var}(W|\theta)$ for all θ.

That is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

Related Theorems

Theorem 7.3.19 - Uniqueness of UMVUE

If W is a best unbiased estimator of $\tau(\theta)$, then W is unique.

Theorem 7.3.20 - UMVUE and unbiased estimators of zero

If $E[W(\mathbf{X})] = \tau(\theta)$. W is the best unbiased estimator of $\tau(\theta)$ if an only if W is uncorrelated with all unbiased estimator of 0.

The power of complete sufficient statistics

Theorem 7.3.23

Let T be a complete sufficient statistic for parameter θ. Let $\phi(T)$ be any estimator based on T. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.

Remarks from previous Theorems - #1

From Rao-Blackwell Theorem, we can always improve an unbiased estimator by conditioning it on a sufficient statistics.

- $W(\mathbf{X})$: unbiased for $\tau(\theta)$.
- $T^*(\mathbf{X})$: sufficient statistic for θ.

$\phi(T) = E[W(\mathbf{X})|T(\mathbf{X})]$ is a better unbiased estimator of $\tau(\theta)$.

Remarks from previous Theorems - #2

In fact, we only need to consider functions of minimal sufficient statistics to find the best unbiased estimator.

Let $T(\mathbf{X})$ be a minimal sufficient, and $T^*(\mathbf{X})$ be a sufficient statistic. Then by definition, there exists a function h that satisfies $T = h(T^*)$.

$$\begin{aligned} E[\phi(T)|T^*] &= E[\phi\{h(T^*)\}|T^*] \\ &= \phi\{h(T^*)\} = \phi(T) \end{aligned}$$

Therefore $\phi(T)$ remains the same after conditioning on any sufficient statistic T^*.

Remarks from previous Theorems - #3

Complete sufficient statistics is a very useful ingredient to obtain a UMVUE.

- $\phi(T)$ is an unbiased estimator for $E[\phi(T)] = \tau(\theta)$.
- By Theorem 7.3.20, $\phi(T)$ is the best unbiased estimator if and only if $\phi(T)$ if and only of $\phi(T)$ is uncorrelated with $U(T)$, which is any unbiased esimator of 0.
- By definition, T is complete is $E[U(T)] = 0$ for all θ implies $U(T) = 0$ almost surely.
- Suppose that T is a complete statistic, then $U(T)$ can only be zero almost surely.
- Therefore, $\text{Cov}(\phi(T), U(T)) = \text{Cov}(\phi(T), 0) = 0$, and $\phi(T)$ is the best unbiased estimator of its expected value (Theorem 7.3.23).

Summary of Method 2 for obtaining UMVUE

Use complete sufficient statistic to find the best unbiased estimator for $\tau(\theta)$.

1. Find complete sufficient statistic T for θ.
2. Obtain $\phi(T)$, an unbiased estimator of $\tau(\theta)$ using either of the following two ways
   - Guess a functon $\phi(T)$ such that $E[\phi(T)] = \tau(\theta)$.
   - Guess an unbiased estimator $h(\mathbf{X})$ of $\tau(\theta)$. Construct $\phi(T) = E[h(\mathbf{X})|T]$, then $E[\phi(T)] = E[h(\mathbf{X})] = \tau(\theta)$.

Example - Normal Distribution

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$. Find the best unbiased estimator for (1) μ, (2) σ^2, (3) μ^2.

Solution

- First, we need to find a complete and sufficient statistic for (μ, σ^2).
- We know that $\mathbf{T}(\mathbf{X}) = (\overline{X}, s^2_{\mathbf{X}})$ is complete, sufficient statistic for (μ, σ^2).
- Because $E[\overline{X}] = \mu$, $\overline{X}$ is an unbiased estimator for μ, $\overline{X}$ is also a function of $\mathbf{T}(\mathbf{X})$.
- Therefore, $\overline{X}$ is the best unbiased estimator for μ.

Example - Normal Distribution (cont'd)

- $E(s^2_{\mathbf{X}}) = \sigma^2$
- $s^2_{\mathbf{X}}$ is a function of $\mathbf{T}$
- Therefore $s^2_{\mathbf{X}}$ is the best unbiased estimator of σ^2.

Example - Normal Distribution (cont'd)

To obtain UMVUE for μ^2, we need a $\phi(\mathbf{T}) = \phi(\overline{X}, s^2_{\mathbf{X}})$ such that $E[\phi(\mathbf{T})] = \mu^2$.

$$\begin{aligned} E(\overline{X}) &= \mu \\ E((\overline{X})^2) &= \text{Var}(\overline{X}) + E(\overline{X}^2) = \frac{\sigma^2}{n} + \mu^2 \\ E\left(\overline{X}^2 - \frac{\sigma^2}{n}\right) &= \mu^2 \\ E\left(\overline{X}^2 - \frac{s^2_{\mathbf{X}}}{n}\right) &= \mu^2 \end{aligned}$$

- $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is unbiased estimator for μ^2
- And it is a function of $(\overline{X}, s^2_{\mathbf{X}})$.
- Hence, $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is the best unbiased estimator for μ^2.

Example - Normal Distribution - Alternative method

X_1X_2 is unbiased for μ^2 because $E[X_1X_2] = E(X_1)E(X_2) = \mu^2$.

$$\begin{aligned}
\phi(T) &= E[X_1X_2|\mathbf{T}] = \frac{\sum_{i\neq j} E[X_iX_j|\mathbf{T}]}{n(n-1)} \\
&= \frac{\sum_{i=1}^n E[X_i^2|\mathbf{T}] + \sum_{i\neq j} E[X_iX_j|\mathbf{T}] - \sum_{i=1}^n E[X_i^2|\mathbf{T}]}{n(n-1)} \\
&= \frac{E[(\sum_{i=1}^n X_i)^2|\mathbf{T}] - E[\sum_{i=1}^n X_i^2|\mathbf{T}]}{n(n-1)} \\
&= \frac{E[(n\overline{X})^2 - (n-1)s_{\mathbf{X}}^2 - n\overline{X}^2|\mathbf{T}]}{n(n-1)} = \frac{n(n-1)\overline{X}^2 - (n-1)s_{\mathbf{X}}^2}{n(n-1)} \\
&= \overline{X}^2 - s_{\mathbf{X}}^2/n
\end{aligned}$$

Example - Uniform Distribution

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0,\theta)$. Find the best unbiased estimator for (1) θ, (2) $g(\theta)$ differentiable on $(0,\theta)$ (3) θ^2, (4) $1/\theta$.

Solution - MVUE of θ

- $T(\mathbf{X}) = X_{(n)}$ is a complete and sufficient statistic for θ.
- $f_T(t) = n\theta^{-n}t^{n-1}I(0 < t < \theta)$.
- $E[T] = E[X_{(n)}] = \int_0^\infty tn\theta^{-n}t^{n-1}dt = \frac{n}{n+1}\theta$ (biased)
- $E[\phi(T)] = E\left[\frac{n+1}{n}X_{(n)}\right] = \theta$.

$\frac{n+1}{n}X_{(n)}$ is the best unbiased estimator of θ.

Example - Uniform Distribution - for $g(\theta)$

We need to find a function of $\phi(T) = X_{(n)}$ such that $E[\phi(T)] = g(\theta)$.

$$g(\theta) = E[\phi(T)] = \int_0^\theta \phi(t)n\theta^{-n}t^{n-1}dt$$

Taking derivative with respect to θ, and applying Leibnitz's rule.

$$\begin{aligned}
g'(\theta) &= \frac{d}{d\theta}\int_0^\theta \phi(t)n\theta^{-n}t^{n-1}dt \\
&= \phi(\theta)n\theta^{-n}\theta^{n-1} + \int_0^\theta \phi(t)t^{n-1}n\frac{d}{d\theta}\theta^{-n}dt \\
&= \phi(\theta)n\theta^{-1} + \int_0^\theta \phi(t)t^{n-1}n(-n)\theta^{-n-1}dt \\
&= \phi(\theta)n\theta^{-1} - n\theta^{-1}\int_0^\theta \phi(t)nt^{n-1}\theta^{-n}dt
\end{aligned}$$

Example - Uniform Distribution - for $g(\theta)$ (cont'd)

$$\begin{aligned}
g'(\theta) &= \phi(\theta)n\theta^{-1} - n\theta^{-1}\int_0^\theta \phi(t)nt^{n-1}\theta^{-n}dt \\
&= \phi(\theta)n\theta^{-1} - n\theta^{-1}E[\phi(T)] \\
&= \phi(\theta)n\theta^{-1} - n\theta^{-1}g(\theta) \\
\phi(\theta) &= \frac{g'(\theta) + n\theta^{-1}g(\theta)}{n\theta^{-1}}
\end{aligned}$$

Therefore, the best unbiased estimator of $g(\theta)$ is

$$\begin{aligned}
\phi(T) &= \frac{g'(T) + nT^{-1}g(T)}{nT^{-1}} \\
\phi(X_{(n)}) &= \frac{g'(X_{(n)}) + nX_{(n)}^{-1}g(X_{(n)})}{nX_{(n)}^{-1}} \\
&= \frac{1}{n}X_{(n)}g'(X_{(n)}) + g(X_{(n)})
\end{aligned}$$

Example - Uniform Distribution - for θ^2

$g(\theta) = \theta^2$, and $g'(\theta) = 2\theta$.

$$\begin{aligned} \phi(X_{(n)}) &= \frac{1}{n} X_{(n)} \cdot 2X_{(n)} + X_{(n)}^2 \\ &= \frac{n+2}{n} X_{(n)}^2 \end{aligned}$$

Example - Uniform Distribution - for $1/\theta$

$g(\theta) = 1/\theta$, and $g'(\theta) = -1/\theta^2$.

$$\begin{aligned} \phi(X_{(n)}) &= \frac{1}{n} X_{(n)} \cdot \left(-\frac{1}{X_{(n)}^2}\right) + \frac{1}{X_{(n)}} \\ &= \frac{n-1}{nX_{(n)}} \end{aligned}$$

Example - Binomial best unbiased estimator

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binomial}(k, \theta)$. Estimate the probability of exactly one success.

Solution

- The quantity we need to estimate is
  $$\tau(\theta) = \Pr(X = 1|\theta) = k\theta(1-\theta)^{k-1}$$
- We know that $T(\mathbf{X}) = \sum_{i=1}^n X_i \sim \text{Binomial}(kn, \theta)$ and it is a complete sufficient statistic.
- So we need to find a $\phi(T)$ that satisfies $E[\phi(T)] = \tau(\theta)$.
- There is no imeediately evident unbiased estimator of $\tau(\theta)$ as a function of T.

Solution - Binomial best unbiased estimator

- Start with a simple-minded estimator
  $$W(\mathbf{X}) = \begin{cases} 1 & X_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$
- The expectation of W is
  $$\begin{aligned} E[W] &= \sum_{x_1=0}^{k} W(x_1) \binom{k}{x_1} \theta^{x_1} (1-\theta)^{k-x_1} \\ &= k\theta(1-\theta)^{k-1} \end{aligned}$$
  and hence is an unbiased estimator of $\tau(\theta) = k\theta(1-\theta)^{k-1}$.
- The best unbiased estimator of $\tau(\theta)$ is
  $$\phi(T) = E[W|T] = E[W(\mathbf{X})|T(\mathbf{X})]$$

Solution - Binomial best unbiased estimator (cont'd)

$$\begin{aligned}\phi(t) &= E\left[W(\mathbf{X})|\sum_{i=1}^n X_i = t\right] = \Pr\left[X_1 = 1|\sum_{i=1}^n X_i = t\right] \\ &= \frac{\Pr(X_1 = 1, \sum_{i=1}^n X_i = t)}{\Pr(\sum_{i=1}^n X_i = t)} \\ &= \frac{\Pr(X_1 = 1, \sum_{i=2}^n X_i = t-1)}{\Pr(\sum_{i=1}^n X_i = t)} \\ &= \frac{\Pr(X_1 = 1)\Pr(\sum_{i=2}^n X_i = t-1)}{\Pr(\sum_{i=1}^n X_i = t)} \\ &= \frac{[k\theta(1-\theta)^{k-1}]\left[\binom{k(n-1)}{t-1}\theta^{t-1}(1-\theta)^{k(n-1)-t-1}\right]}{\binom{kn}{n}\theta^t(1-\theta)^{kn-t}} = k\frac{\binom{k(n-1)}{t-1}}{\binom{kn}{t}}\end{aligned}$$

Solution - Binomial best unbiased estimator (cont'd

Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

$$\phi\left(\sum_{i=1}^n X_i\right) = k\frac{\binom{k(n-1)}{\sum X_i - 1}}{\binom{kn}{\sum X_i}}$$

Summary

Today

- Rao-Blackwell Theorem
- Methods for obtaining UMVUE

Next Lecture

- Bayesian Estimators