Biostatistics 615/815 Lecture 9: Dynamic Programming

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Recap: Hash Tables

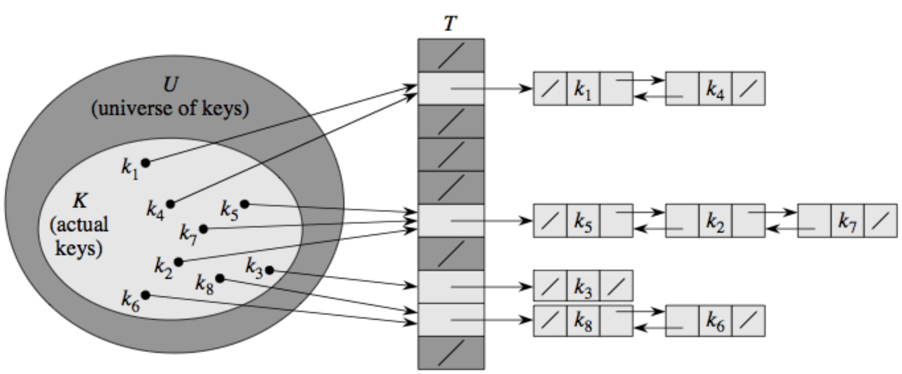
Key features

- $\Theta(1)$ complexity for INSERT, SEARCH, and REMOVE
- Requires large memory space than the actual content for maintainng good performance
- But uses much smaller memory than direct-addres tables

Key components

- Hash function
  - $h(x.key)$ mapping key onto smaller 'addressible' space H
  - Total required memory is the possible number of hash values
  - Good hash function minimize the possibility of key collisions
- Collision-resolution strategy, when $h(k_1) = h(k_2)$.

Recap: Illustration of CHAINEDHASH



Recap : Open hash

Probing strategies

- Linear probing
- Quadratic probing
- Double hashing

Double Hashing

- $h(k, i) = (h_1(k) + ih_2(k)) \mod m$
- The probe sequence depends in two ways upon k.
- For example, $h_1(k) = k \mod m$, $h_2(k) = 1 + (k \mod m')$
- Avoid clustering problem
- Performance close to ideal scheme of uniform hashing.

Today

Dynamic Programming

- Fibonacci numbers
- Manhattan tourist problems
- Edit distance problem

Recap: Divide and conquer algorithms

Good examples of divide and conquer algorithms

- TowerOfHanoi
- MergeSort
- QuickSort
- BinarySearchTree algorithms

These algorithms divide a problem into smaller and disjoint subproblems until they become trivial.

A divide-and-conquer algorithms for Fibonacci numbers

Fibonacci numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & n > 1 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases}$$

A recursive implementation of fibonacci numbers

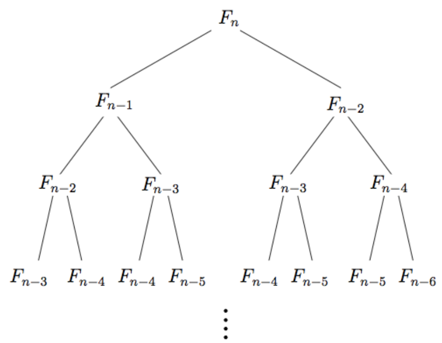
```
int fibonacci(int n) {
  if ( n < 2 ) return n;
  else return fibonacci(n-1)+fibonacci(n-2);
}
```


Performance of recursive Fibonacci

Computational time

- 4.4 seconds for calculating F_{40}
- 49 seconds for calculating F_{45}
- ∞ seconds for calculating F_{100}!

What is happening in the recursive FIBONACCI



Time complexity of redundant FIBONACCI

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) \\ T(1) &= 1 \\ T(0) &= 1 \\ T(n) &= F_{n+1} \end{aligned}$$

The time complexity is exponential

A non-redundant FIBONACCI

```
int fibonacci(int n) {
  int* fibs = new int[n+1];
  fibs[0] = 0;
  fibs[1] = 1;
  for(int i=2; i <= n; ++i) {
    fibs[i] = fibs[i-1]+fibs[i-2];
  }
  int ret = fibs[n];
  delete [] fibs;
  return ret;
}
```


Key idea in non-redundant FIBONACCI

- Each F_n will be reused to calculate F_{n+1} and F_{n+2}
- Store F_n into an array so that we don't have to recalculate it

A recursive, but non-redundant FIBONACCI

```
int fibonacci(int* fibs, int n) {
  if ( fibs[n] > 0 ) {
    return fibs[n];    // reuse stored solution if available
  }
  else if ( n < 2 ) {
    return n;          // terminal condition
  }
  fibs[n] = fibonacci(n-1) + fibonacci(n-2); // store the solution once computed
  return fibs[n];
}
```


Dynamic programming

Key components of dynamic programing

- Problems that can be divided into subproblems
- Overlapping subproblems - subproblems share subsubproblems
- Solves each subsubproblem just once and then saves its answer

Why *dynamic* programming?

According to wikipedia... *"The word 'dynamic' was chosen because it sounded impressive, not because how the method works"*

Examples of dynamic programming

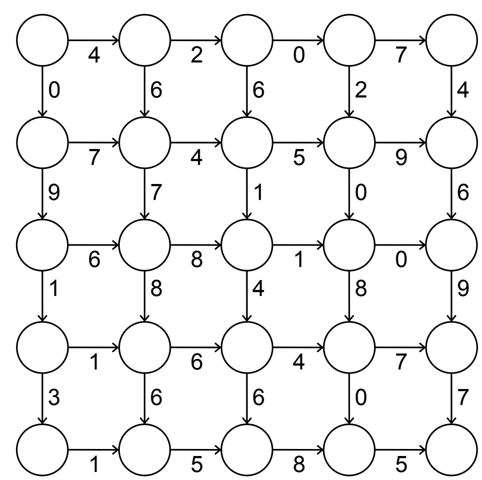
- Shortest path finding algorithms
- DNA sequence alignment
- Hidden markov models

Steps of dynamic programming

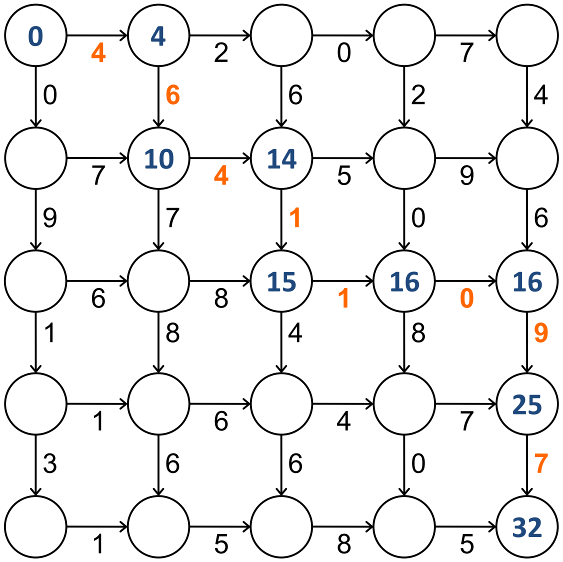
- Characterize the structure of an (optimal) solution
- Recursively define the value of an (optimal) solution
- Compute the value of an (optimal) solution, typically in a bottom-up fashion
- Construct an optimal solution from computed information.

The Manhattan tourist problem

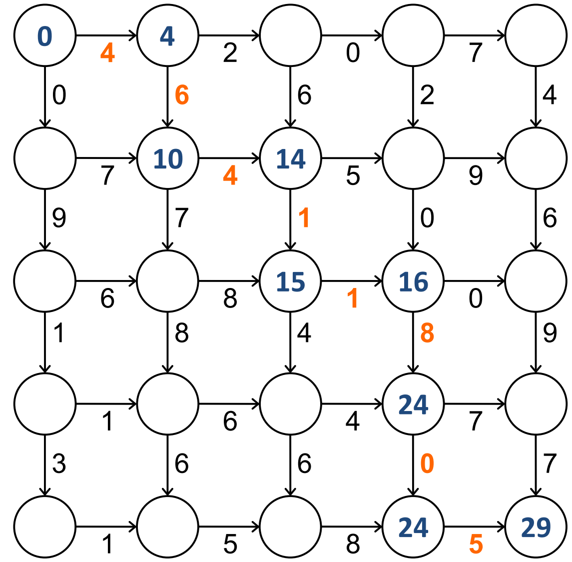
Find the cost-optimal path from left-top corner to right-bottom corner



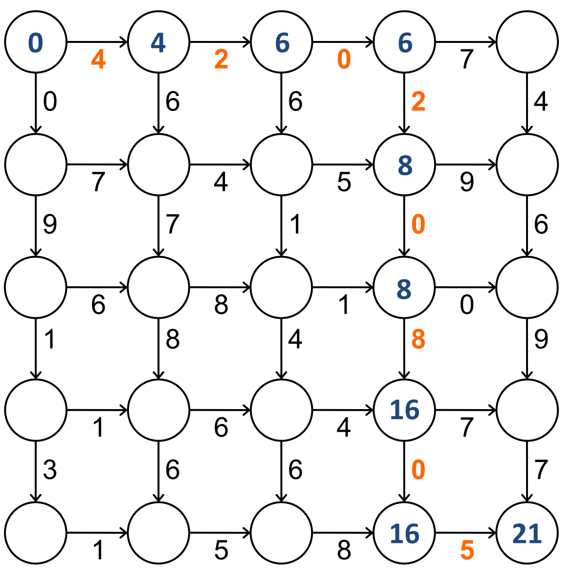
One possible (but not optimal) solution



A slightly better, but still not an optimal solution



And here comes an optimal solution



A brute-force algorithm

Algorithm BruteForceMTP

1. Enumerate all the possible paths
2. Calculate the cost of each possible path
3. Pick the path that produces a minimum cost

Time complexity

- Number of possible paths are $\binom{n_r+n_c}{n_r}$
- Super-exponential growth when n_r and n_c are similar.

A "dynamic" structure of the solution

- Let $C(r, c)$ be the optimal cost from $(0, 0)$ to (r, c)
- Let $h(r, c)$ be the weight from (r, c) to $(r, c + 1)$
- Let $v(r, c)$ be the weight from (r, c) to $(r + 1, c)$
- We can recursively define the optimal cost as

$$C(r,c) = \begin{cases} \min \begin{cases} C(r-1,c) + v(r-1,c) \\ C(r,c-1) + h(r,c-1) \end{cases} & r > 0, c > 0 \\ C(r,c-1) + h(r,c-1) & r > 0, c = 0 \\ C(r-1,c) + v(r-1,c) & r = 0, c > 0 \\ 0 & r = 0, c = 0 \end{cases}$$

- Once $C(r, c)$ is evaluated, it must be stored to avoid redundant computation.

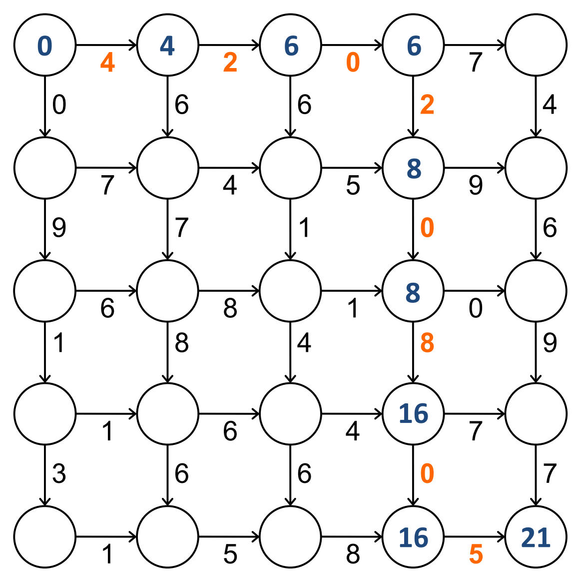
Time complexity of the "dynamic" solution

- Each recursive step takes a constant time
- Each $C(r, c)$ is evaluated at most once.
- Total time complexity is $\Theta(n_r n_c)$.
- Like Fibonacci search, the time complexity would be super exponential if $C(r, c)$ is not stored and redundantly evaluated.

Reconstructing the optimal path

- Optimal cost does not automatically produce optimal path.
- When choosing smaller-cost path between two alternatives, store the decision
- Backtrack from the destination to the source based on the stored decision

Example of backtracking the path



Implementing Manhattan tourist algorithm

```
template<class T>
class Matrix {  // Matrix data type to store the costs
  T* data;      // internal data as one-dimensional array
  int nr, nc;   // # rows and # cols
  Matrix(const Matrix<T>& m) {};   // prevent copy
public:
  Matrix(int nrows, int ncols) : nr(nrows), nc(ncols) {
    data = new T[nrows*ncols]();   // initialize matrix
  }
  ~Matrix() {
    if ( data != NULL ) delete [] data;
  }
  // accessor function : possible to use to read/write elements
  // value1 = M.at(i,j);
  // M.at(i,j) = value2;
  T& at(int r, int c) { return data[r*nc+c]; }
  void print(); // print the content of the matrix (omitted)
};
```


Manhattan tourist problem : main()

```
int main(int argc, char** argv) {
  int nrows = 5, ncols = 5;
  Matrix<int> hw(nrows,ncols-1), vw(nrows-1,ncols); // weight matrices
  hw.at(0,0) = 4; hw.at(0,1) = 2; ... // initialize horizontal weights
  vw.at(0,0) = 0; vw.at(0,1) = 6; ... // initialize vertical weights

  // optimal costs and decisions for backtracking
  Matrix<int> cost(nrows,ncols), move(nrows,ncols);
  // calculate the optimal cost, recording the backtracking info
  int optCost = optimalCost(hw,vw,cost,move,nrows-1,ncols-1);
  std::cout << "Optimal cost is " << optCost << std::endl;
  // backtrack the stored decision to reconstruct an optimal path
  trackOptimalPath(hw,vw,cost,move,nrows-1,ncols-1);
  return 0;
}
```


Calculating optimal cost

```
// hw, vw : horizontal and vertical input weights
// cost : stored optimal cost from (0,0) to (r,c)
// move : stored optimal decision to reach (r,c)
// r,c  : the position of interest
int optimalCost(Matrix<int>& hw, Matrix<int>& vw,
         Matrix<int>& cost, Matrix<int>& move, int r, int c) {
  // if cost is stored already, skip the cost evaluation
  if ( cost.at(r,c) == 0 ) {
    if ( ( r == 0 ) && ( c == 0 ) ) cost.at(r,c) = 0; // terminal condition
    else if ( r == 0 ) { // only horizontal move is possible
      move.at(r,c) = 0;  // 0 means horitontal move to (r,c)
      cost.at(r,c) = optimalCost(hw,vw,cost,move,r,c-1) + hw.at(r,c-1);
    }
    else if ( c == 0 ) { // only vertical move is possible
      move.at(r,c) = 1;  // 1 means vertical move to (r,c)
      cost.at(r,c) = optimalCost(hw,vw,cost,move,r-1,c) + vw.at(r-1,c);
    }
```


Calculating optimal cost (cont'd)

```
    else { // evaluate the cumulative cost of horizontal and vertical move
      int hcost = optimalCost(hw,vw,cost,move,r,c-1) + hw.at(r,c-1);
      int vcost = optimalCost(hw,vw,cost,move,r-1,c) + vw.at(r-1,c);
      if ( hcost > vcost ) {   // when vertical move is optimal
        move.at(r,c) = 1;      // store the decision
        cost.at(r,c) = vcost;  // and store the optimal cost
      }
      else {                   // when horizontal move is optimal
        move.at(r,c) = 0;
        cost.at(r,c) = hcost;
      }
    }
  }
  return cost.at(r,c); // return the optimal cost
}
```

Dynamic programming : A smart recursion

- Dynamic programming is recursion without repetition
  1. Formulate the problem recursively
  2. Build solutions to your recurrence from the bottom up
- Dynamic programming is not about filling in tables; it's about smart recursion (Jeff Erickson)

Minimum edit distance problem

Edit distance

Minimum number of letter insertions, deletions, substitutions required to transform one word into another

An example

F̲OOD → MOO̲D → MON^D → MONE̲D → MONEY

Edit distance is 4 in the example above

More examples of edit distance

```
        F   O   O       D
        M   O   N   E   Y

A   L   G   O   R       I       T   H   M
A   L       T   R   U   I   S   T   I   C
```

- Similar representation to DNA sequence alignment
- Does the above alignment provides an optimal edit distance?

A dynamic programming solution

		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	**0**	1	2	3	4	5	6	7	8
L	2	1	**0**	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	**4**	5	6
R	4	3	2	2	2	**2**	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	**3**	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	**4**	5	6
I	9	8	7	7	7	7	**6**	5	5	6
C	10	9	8	8	8	8	7	6	6	6

Recursively formulating the problem

- Input strings are $x[1, \cdots, m]$ and $y[1, \cdots, n]$.
- Let $x_i = x[1, \cdots, i]$ and $y_j = y[1, \cdots, j]$ be substrings of x and y.
- Edit distance $d(x, y)$ can be recursively defined as follows

$$d(x_i, y_j) = \begin{cases} i & j = 0 \\ j & i = 0 \\ \min \left\{ \begin{array}{l} d(x_{i-1}, y_j) + 1 \\ d(x_i, y_{j-1}) + 1 \\ d(x_{i-1}, y_{i-1}) + I(x[i] \neq y[j]) \end{array} \right\} & otherwise \end{cases}$$

- Similar to the Manhattan tourist problem, but with 3-way choice.
- Time complexity is $\Theta(mn)$.

Summary

Today

- Dynamic programming is a smart recursion avoiding redundancy
- Divide a problem into subproblems that can be shared
- Examples of dynamic programming
  - Fibonacci numbers
  - Manhattan tourist problem
  - Edit distance problem

Next lecture

- Algorithms in graphs
  - Using `boost` library
  - Dijkstra's algorithm (CLRS Chapter 24)
- Introduction to hidden Markov model