Biostatistics 602 - Statistical Inference
Lecture 13
Rao-Blackwell Theorem

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February 26th, 2013

Last Lecture

Submit your answers (after the question ID) either

- At http://pollEv.com
- By text to 22333

117261 Which family of distribution is always guaranteed to satisfy the interchangeability condition?

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HandsUp If the Cramer-Rao bound is not attainable, does it imply that the estimator cannot be UMVUE?

Recap - Using Leibnitz's Rule

Leibnitz's Rule

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x|\theta)\,dx = f(b(\theta)|\theta)\,b'(\theta) - f(a(\theta)|\theta)\,a'(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x|\theta)\,dx$$

Applying to Uniform Distribution

$$\begin{aligned} f_X(x|\theta) &= 1/\theta \\ \frac{d}{d\theta}\int_0^{\theta} h(x)\left(\frac{1}{\theta}\right)dx &= \frac{h(\theta)}{\theta}\frac{d\theta}{d\theta} - h(0)f_X(0|\theta)\frac{d0}{d\theta} + \int_0^{\theta}\frac{\partial}{\partial\theta}h(x)\left(\frac{1}{\theta}\right)dx \\ &\neq \int_0^{\theta}\frac{\partial}{\partial\theta}h(x)\left(\frac{1}{\theta}\right)dx \end{aligned}$$

The interchangeability condition is not satisfied.

Recap - When is the Cramer-Rao Lower Bound Attainable?

It is possible that the value of Cramer-Rao bound may be strictly smaller than the variance of any unbiased estimator

Corollary 7.3.15 : Attainment of Cramer-Rao Bound

Let $X_1, \cdots, X_n$ be iid with pdf/pmf $f_X(x|\theta)$, where $f_X(x|\theta)$ satisfies the assumptions of the Cramer-Rao Theorem.
Let $L(\theta|\mathbf{x}) = \prod_{i=1}^{n} f_X(x_i|\theta)$ denote the likelihood function. If $W(\mathbf{X})$ is unbiased for $\tau(\theta)$, then $W(\mathbf{X})$ attains the Cramer-Rao lower bound if and only if

$$\frac{\partial}{\partial\theta} \log L(\theta|\mathbf{x}) = S_n(\mathbf{x}|\theta) = a(\theta)[W(\mathbf{X}) - t(\theta)]$$

for some function $a(\theta)$.

Recap - Attainability of C-R bound for σ^2 in $\mathcal{N}(\mu, \sigma^2)$

1. If μ is known, the best unbiased estimator for σ^2 is $\sum_{i=1}^n (x_i - \mu)^2/n$, and it attains the Cramer-Rao lower bound, i.e.

$$\text{Var}\left[\frac{\sum_{i=1}^n (X_i - \mu)^2}{n}\right] = \frac{2\sigma^4}{n}$$

2. If μ is not known, the Cramer-Rao lower-bound cannot be attained.

At this point, we do not know if $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2$ is the best unbiased estimator for σ^2 or not.

Fact for one-parameter exponential family

Let $X_1, \cdots, X_n$ be iid from the one parameter exponential family with pdf/pmf $f_X(x|\theta) = c(\theta)h(x)\exp\left[w(\theta)t(x)\right]$.

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Assume that $E[t(X)] = \tau(\theta)$. Then $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$, which is an unbiased estimator of $\tau(\theta)$, attains the Cramer-Rao lower-bound.

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Assume that $E[t(X)] = \tau(\theta)$. Then $\frac{1}{n}\sum_{i=1}^n t(x_i)$, which is an unbiased estimator of $\tau(\theta)$, attains the Cramer-Rao lower-bound. That is,

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^n t(X_i)\right) = \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

Proof

$$E\left[\frac{1}{n}\sum_{i=1}^{n} t(X_i)\right] \quad = \quad E[t(X_1)] = \cdots = E[t(X_n)] = \tau(\theta)$$

Proof

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So, $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$ is an unbiased estimator of $\tau(\theta)$.

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So, $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$ is an unbiased estimator of $\tau(\theta)$.

$$\begin{aligned}\log L(\theta|\mathbf{x}) &= \sum_{i=1}^{n} \log f_X(x_i|\theta) \\ &= \sum_{i=1}^{n} [\log c(\theta) + \log h(x) + w(\theta)t(x_i)]\end{aligned}$$

Proof (cont'd)

$$\frac{\partial \log L(\theta|\mathbf{x})}{\partial \theta} = \sum_{i=1}^{n} \left[\frac{c'(\theta)}{c(\theta)} + 0 + w'(\theta) t(x_i) \right]$$

Proof (cont'd)

$$\begin{aligned}\frac{\partial \log L(\theta|\mathbf{x})}{\partial \theta} &= \sum_{i=1}^{n}\left[\frac{c'(\theta)}{c(\theta)} + 0 + w'(\theta)t(x_i)\right] \\ &= nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^{n} t(x_i) - \left\{-\frac{c'(\theta)}{c(\theta)w'(\theta)}\right\}\right]\end{aligned}$$

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- $\frac{1}{n}\sum_{i=1}^n t(x_i)$ is the best unbiased estimator of $-\frac{c'(\theta)}{c(\theta)w'(\theta)}$

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- And it attains the Cramer-Rao lower bound.

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- $\frac{1}{n}\sum_{i=1}^{n} t(x_i)$ is the best unbiased estimator of $-\frac{c'(\theta)}{c(\theta)w'(\theta)}$
- And it attains the Cramer-Rao lower bound.
- Because $E\left[\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x})\right] = 0$, $\tau(\theta) = -\frac{c'(\theta)}{c(\theta)w'(\theta)}$.

Cramer-Rao Theorem on Exponential Family

Fact

$$f_X(x|\theta) = c(\theta)h(x)\exp\left[w(\theta)t(x)\right]$$

If $X_1, \cdots, X_n$ are iid samples from $f_X(x|\theta)$, $\frac{1}{n}\sum_{i=1}^{n} t(X_i)$ is the best unbiased estimator for its expected value.

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$$E[t(X)] = \tau(\theta)$$

$$\text{Var}\left[\frac{1}{n}\sum_{i=1}^n t(X_i)\right] = \frac{[\tau'(\theta)]^2}{I_n(\theta)}$$

Proof

$$\frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}) \quad = \quad n w'(\theta) \left[\frac{1}{n} \sum_{i=1}^{n} t(X_i) + \frac{c'(\theta)}{c(\theta) w'(\theta)} \right]$$

Proof

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\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) &= nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^{n} t(X_i) + \frac{c'(\theta)}{c(\theta)w'(\theta)}\right] \\
\tau(\theta) &= -\frac{c'(\theta)}{c(\theta)w'(\theta)}
\end{aligned}$$

Proof

$$\frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}) = nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^{n} t(X_i) + \frac{c'(\theta)}{c(\theta)w'(\theta)}\right]$$

$$\tau(\theta) = -\frac{c'(\theta)}{c(\theta)w'(\theta)}$$

$$\frac{\partial}{\partial \theta} \log L(\theta|\mathbf{x}) = a(\theta)[W(\mathbf{x}) - \tau(\theta)]$$

where $a(\theta) = nw'(\theta)$, $W(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} t(x_i)$

Obtaining $I_n(\theta)$

$$\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x}) = nw'(\theta)\left[\frac{1}{n}\sum_{i=1}^n t(X_i) - \tau(\theta)\right]$$

Obtaining $I_n(\theta)$

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$$E\left[\left\{\frac{\partial}{\partial\theta}\log L(\theta|\mathbf{x})\right\}^2\right] = I_n(\theta) = E\left[(nw'(\theta))^2\left(\frac{1}{n}\sum_{i=1}^{n} t(X_i) - \tau(\theta)\right)^2\right]$$

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&= \mathrm{Var}\left[nw'(\theta)\left\{\frac{1}{n}\sum_{i=1}^{n} t(X_i) - \tau(\theta)\right\}\right] \\
&= n^2\left\{w'(\theta)\right\}^2 \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} t(X_i)\right]
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&= n^2\left\{w'(\theta)\right\}^2 \frac{[\tau'(\theta)]^2}{I_n(\theta)}
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\left[nw'(\theta)\right]^2 &= \frac{I_n(\theta)\cdot I_n(\theta)}{[\tau'(\theta)]^2} \\
&= \left(\frac{I_n(\theta)}{\tau'(\theta)}\right)^2 \\
I_n(\theta) &= |nw'(\theta)\tau'(\theta)|
\end{aligned}
$$

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    - It helps to confirm an estimator is the best unbiased estimator of $\tau(\theta)$ if it happens to attain the CR-bound.
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2. When "regularity conditions" are not satisfied, $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$ is no longer a valid lower bound.

Summary

1. If "regularity conditions" are satisfied, then we have a Cramer-Rao bound for unbiased estimators of $\tau(\theta)$.
   - It helps to confirm an estimator is the best unbiased estimator of $\tau(\theta)$ if it happens to attain the CR-bound.
   - If an unbiased estimator of $\tau(\theta)$ has variance greater than the CR-bound, it does NOT mean that it is not the best unbiased estimator.
2. When "regularity conditions" are not satisfied, $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$ is no longer a valid lower bound.
   - There may be unbiased estimators of $\tau(\theta)$ that have variance smaller than $\frac{[\tau'(\theta)]^2}{I_n(\theta)}$.

Methods for finding best unbiased estimator

1. Using Cramer-Rao bound

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2. Using Rao-Blackwell theorem
   - Use complete and sufficient statistic.
   - Find a 'better' unbiased estimator

Important Facts

X and Y are two random variables

- $E(X) = E[E(X|Y)]$ (Theorem 4.4.3)

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- $\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}[E(X|Y)]$ (Theorem 4.4.7)
- $E[g(X)|Y] = \int_{x \in \mathcal{X}} g(x)f(x|Y)\,dx$ is a function of Y.
- If X and Y are independent, $E[g(X)|Y] = E[g(X)]$.

Seeking for a better unbiased estimator

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$$\begin{aligned} \phi(T) &= E(W(\mathbf{X})|T) \\ E[\phi(T)] &= E[E(W(\mathbf{X})|T)] = E[W(\mathbf{X})] = \tau(\theta) \qquad \text{(unbiased for } \tau(\theta)) \end{aligned}$$

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\mathrm{Var}(\phi(T)) &= \mathrm{Var}[E(W|T)] \\
&= \mathrm{Var}(W) - E[\mathrm{Var}(W|T)] \\
&\leq \mathrm{Var}(W) \qquad \text{(smaller variance than } W)
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1. If $\phi(T)$ is an estimator, then $\phi(T)$ is equal or better than $W(\mathbf{X})$.
2. $\phi(T) = E[W|T] = E[W|T,\theta]$.

$\phi(T)$ may depend on θ, which means that $\phi(T)$ may not be an estimator.

Example 1

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$. $W(\mathbf{X}) = \frac{1}{2}(X_1 + X_2)$ is an unbiased estimator of θ.

Consider conditioning it on $T(\mathbf{X}) = X_1$.

$$\phi(T) = E[W|T] = E\left[\frac{1}{2}(X_1 + X_2)|X_1\right]$$

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&= \frac{1}{2}X_1 + \frac{1}{2}E(X_2) \\
&= \frac{1}{2}X_1 + \frac{1}{2}\theta
\end{aligned}$$

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- $\text{Var}[\phi(T)] = \frac{\text{Var}(X)}{n} = \frac{1}{n} < \text{Var}(W) = 1$
- $\phi(T)$ is an estimator.

Rao-Blackwell Theorem

Theorem 7.3.17

Let $W(\mathbf{X})$ be any unbiased estimator of $\tau(\theta)$, and T be a sufficient statistic for θ.

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2. $\text{Var}[\phi(T)|\theta] \leq \text{Var}(W|\theta)$ for all θ.

That is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

Proof of Rao-Blackwell Theorem

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Uniqueness of UMVUE

Theorem 7.3.19

If W is a best unbiased estimator of $\tau(\theta)$, then W is unique.

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E(W_2) &= a\tau(\theta) + b \\
&= \tau(\theta)
\end{aligned}$$

$a = 1, b = 0$ must hold, and $W_2 = W_1$. Therefore, the best unbiased estimator is unique.

Unbiased estimator of zero

Definition

If $U(\mathbf{X})$ satisfies $E(U) = 0$. Then we call U an unbiased estimator of 0.

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Theorem 7.3.20

If $E[W(\mathbf{X})] = \tau(\theta)$. W is the best unbiased estimator of $\tau(\theta)$ if an only if W is uncorrelated with all unbiased estimator of 0.

Proof of Theorem 7.3.20

Let W be an unbiased estimator of $\tau(\theta)$. Let $V = W + U$ and $U \in \mathcal{U}$, which is the class of unbiased estimators of 0.

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$$\mathcal{V} = \{V_a = W + aU\}$$

where a is a constant.

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$$E(V_a) = E(W + aU) = E(W) + aE(U)$$

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$$\begin{aligned} E(V_a) &= E(W + aU) = E(W) + aE(U) \\ &= \tau(\theta) + a \cdot 0 = \tau(\theta) \\ \mathrm{Var}(V_a) &= \mathrm{Var}(W + aU) \end{aligned}$$

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By construction, V is an unbiased estimator of $\tau(\theta)$. Consider

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$$\begin{aligned} E(V_a) &= E(W + aU) = E(W) + aE(U) \\ &= \tau(\theta) + a \cdot 0 = \tau(\theta) \\ \mathrm{Var}(V_a) &= \mathrm{Var}(W + aU) \\ &= a^2\mathrm{Var}(U) + 2a\mathrm{Cov}(W, U) + \mathrm{Var}(W) \end{aligned}$$

Proof of Theorem 7.3.20 (cont'd)

The variance is minimized when

$$a = \frac{-2\text{Cov}(W, U)}{2\text{Var}(U)} = -\frac{\text{Cov}(W, U)}{\text{Var}(U)}$$

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The best unbiased estimator in this class is

$$W - \frac{\text{Cov}(W, U)}{\text{Var}(U)} U$$

Proof of Theorem 7.3.20 (cont'd)

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Proof of Theorem 7.3.20 (cont'd)

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W is the best unbiased estimator in this class if and only if $\mathrm{Cov}(W,U) = 0$. Therefore for W is the best among all unbiased estimators of $\tau(\theta)$ if and only if $\mathrm{Cov}(W,U) = 0$ for every $U \in \mathcal{U}$.

Summary

Today

- Cramer-Rao Theorem with single parameter exponential family.
- Rao-Blackwell Theorem

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- Cramer-Rao Theorem with single parameter exponential family.
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Next Lecture

- More Rao-Blackwell Theorem