Biostatistics 615/815 Lecture 8: Dynamic Programming

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September 27th, 2012

Removing an element from a list

myList.h

```
template <class T>
bool myList<T>::remove(const T& x) {
  if ( head == NULL )
    return false;     // NOT_FOUND if the list is empty
  else {
    // call head->remove will return the object to be removed
    myListNode<T>* p = head->remove(x, head);
    if ( p == NULL ) { // if NOT_FOUND return false
      return false;
    }
    else {              // if FOUND, delete the object before returning true
      delete p;
      return true;
    }
  }
}
```


Removing an element from a list

myListNode.h

```
template <class T>
// pass the pointer to [prevElement->next] so that we can change it
myListNode<T>* myListNode<T>::remove(const T& x, myListNode<T>*& prevNext) {
  if ( value == x ) {   // if FOUND
    prevNext = next;    // *pPrevNext was this, but change to next
    next = NULL;        // disconnect the current object from the list
    return this;        // and return it so that it can be destroyed
  }
  else if ( next == NULL ) {
    return NULL;        // return NULL if NOT_FOUND
  }
  else {
    return next->remove(x, next); // recursively call on the next element
  }
}
```


Key algorithms

REMOVE$(node, x)$

1. If $node.key == x$
   1. If the node is leaf, remove the node
   2. If the node only has left child, replace the current node to the left child
   3. If the node only has right child, replace the current node to the right child
   4. Otherwise, pick either maximum among left sub-tree or minimum among right subtree and substitute the node into the current node
2. If $x < node.key$
   1. Call REMOVE$(node.left, x)$ if $node.left$ exists
   2. Otherwise, return NOTFOUND
3. If $x > node.key$
   1. Call REMOVE$(node.right, x)$ if $node.right$ exists
   2. Otherwise, return NOTFOUND

Binary search tree : REMOVE

myTree.h

```
template <class T>
myTreeNode<T>* myTreeNode<T>::remove(const T& x, myTreeNode<T>*& pSelf) {
  if ( x == value ) {  // key was found
    if ( ( left == NULL ) && ( right == NULL ) ) { // no child
      pSelf = NULL;
      return this;
    }
    else if ( left == NULL ) { // only left is NULL
      pSelf = right;
      right = NULL;
      return this;
    }
    else if ( right == NULL ) {  // only right is NULL
      pSelf = left;
      left = NULL;
      return this;
    } // ....
```


Binary search tree : REMOVE

myTreeNode.h

```
    else { // neither left nor right is NULL
      // choose which subtree to delete
      myTreeNode<T>* p;
      const T& l = left->getMax();
      const T& r = right->getMin();
      if ( value - l < r - value ) { // replace with closer value
        p = left->remove(l, left);
        value = l;
      }
      else {
        p = right->remove(r, right);
        value = r;
      }
      return p;
    }
  }
```


Binary search tree : REMOVE

myTreeNode.h

```
  else if ( x < value ) {
    if ( left == NULL )
      return NULL;
    else
      return left->remove(x, left);
  }
  else { // x > value
    if ( right == NULL )
      return NULL;
    else
      return right->remove(x, right);
  }
}
```


Binary search tree : GETMAX and GETMIN

myTreeNode.h

```
template <class T>
const T& myTreeNode<T>::getMax() {  // return the largest value
  if ( right == NULL ) return value;
  else return right->getMax();
}

template <class T>
const T& myTreeNode<T>::getMin() {  // return the smallest value
  if ( left == NULL ) return value;
  else return left->getMin();
}
```

If you want to print a tree...

myTreeNode.h

```
template <class T> void myTreeNode<T>::print() {
  std::cout << "[ ";
  if ( left != NULL ) left->print();
  else std::cout << "NIL";
  std::cout << " , (" << value << "," << size << ") , ";
  if ( right != NULL ) right->print();
  else std::cout << "NIL";
  std::cout << " ]";
}
```

myTree.h

```
template <class T> void myTree<T>::print() {
  if ( pRoot != NULL ) pRoot->print();
  else std::cout << "(EMPTY TREE)";
  std::cout << std::endl;
}
```


Summary - Binary Search Tree

- Key Features
  - Fast insertion, search, and removal
  - Implementation is much more complicated
- Class Structure
  - `myTree` class to keep the root node
  - `myTreeNode` class to store key and up to two children
- Key Algorithms
  - Insert : Traverse the tree in sorted order and create a new node in the first leaf node.
  - Search : Divide-and-conquer algorithms
  - Remove : Move the nearest leaf element among the subtree and destroy it.

Recap: Divide and conquer algorithms

Good examples of divide and conquer algorithms

- TowerOfHanoi
- MergeSort
- QuickSort
- BinarySearchTree algorithms

These algorithms divide a problem into smaller and disjoint subproblems until they become trivial.

A divide-and-conquer algorithms for Fibonacci numbers

Fibonacci numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & n > 1 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases}$$

A recursive implementation of fibonacci numbers

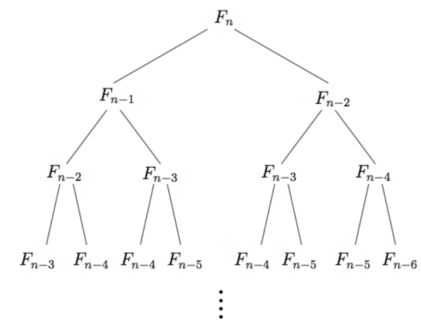
```
int fibonacci(int n) {
  if ( n < 2 ) return n;
  else return fibonacci(n-1)+fibonacci(n-2);
}
```

Performance of recursive FIBONACCI

Computational time

- 4.4 seconds for calculating F_{40}
- 49 seconds for calculating F_{45}
- ∞ seconds for calculating F_{100}!

What is happening in the recursive FIBONACCI



Time complexity of redundant FIBONACCI

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) \\ T(1) &= 1 \\ T(0) &= 1 \\ T(n) &= F_{n+1} \end{aligned}$$

The time complexity is exponential

A non-redundant FIBONACCI

```
int fibonacci(int n) {
  int* fibs = new int[n+1];
  fibs[0] = 0;
  fibs[1] = 1;
  for(int i=2; i <= n; ++i) {
    fibs[i] = fibs[i-1]+fibs[i-2];
  }
  int ret = fibs[n];
  delete [] fibs;
  return ret;
}
```

Key idea in non-redundant FIBONACCI

- Each F_n will be reused to calculate F_{n+1} and F_{n+2}
- Store F_n into an array so that we don't have to recalculate it

A recursive, but non-redundant FIBONACCI

```
int fibonacci(int* fibs, int n) {
  if ( fibs[n] > 0 ) {
    return fibs[n];    // reuse stored solution if available
  }
  else if ( n < 2 ) {
    return n;          // terminal condition
  }
  fibs[n] = fibonacci(n-1) + fibonacci(n-2); // store the solution once computed
  return fibs[n];
}
```


Dynamic programming

Key components of dynamic programing

- Problems that can be divided into subproblems
- Overlapping subproblems - subproblems share subsubproblems
- Solves each subsubproblem just once and then saves its answer

Why *dynamic* programming?

According to wikipedia... *"The word 'dynamic' was chosen because it sounded impressive, not because how the method works"*

Examples of dynamic programming

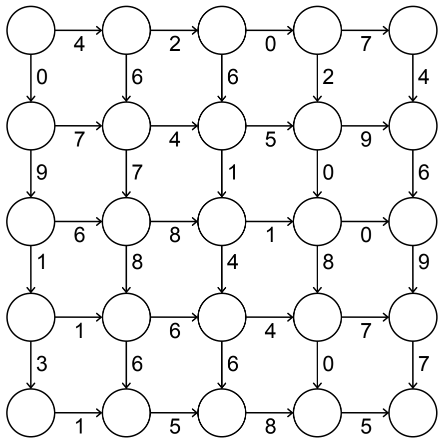
- Shortest path finding algorithms
- DNA sequence alignment
- Hidden markov models

Steps of dynamic programming

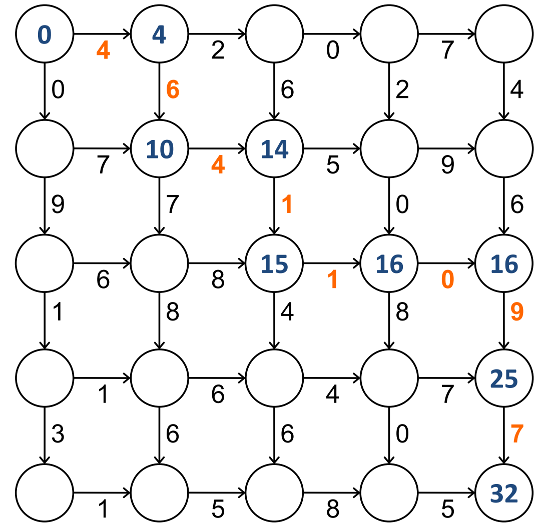
- Characterize the structure of an (optimal) solution
- Recursively define the value of an (optimal) solution
- Compute the value of an (optimal) solution, typically in a bottom-up fashion
- Construct an optimal solution from computed information.

The Manhattan tourist problem

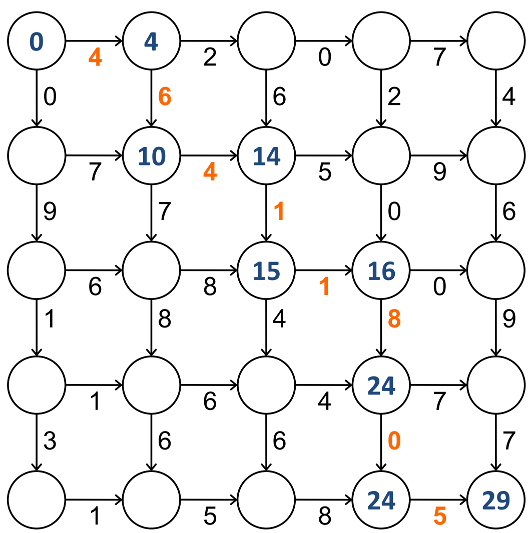
Find the cost-optimal path from left-top corner to right-bottom corner



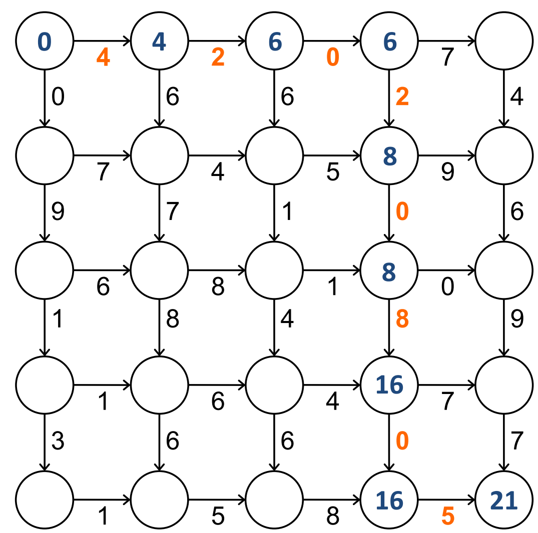
One possible (but not optimal) solution



A slightly better, but still not an optimal solution



And here comes an optimal solution



A brute-force algorithm

Algorithm BruteForceMTP

1. Enumerate all the possible paths
2. Calculate the cost of each possible path
3. Pick the path that produces a minimum cost

Time complexity

- Number of possible paths are $\binom{n_r+n_c}{n_r}$
- Super-exponential growth when n_r and n_c are similar.

A "dynamic" structure of the solution

- Let $C(r, c)$ be the optimal cost from $(0, 0)$ to (r, c)
- Let $h(r, c)$ be the weight from (r, c) to $(r, c + 1)$
- Let $v(r, c)$ be the weight from (r, c) to $(r + 1, c)$
- We can recursively define the optimal cost as

$$C(r,c) = \begin{cases} \min \begin{cases} C(r-1,c) + v(r-1,c) \\ C(r,c-1) + h(r,c-1) \end{cases} & r > 0, c > 0 \\ C(r,c-1) + h(r,c-1) & r = 0, c > 0 \\ C(r-1,c) + v(r-1,c) & r > 0, c = 0 \\ 0 & r = 0, c = 0 \end{cases}$$

- Once $C(r, c)$ is evaluated, it must be stored to avoid redundant computation.

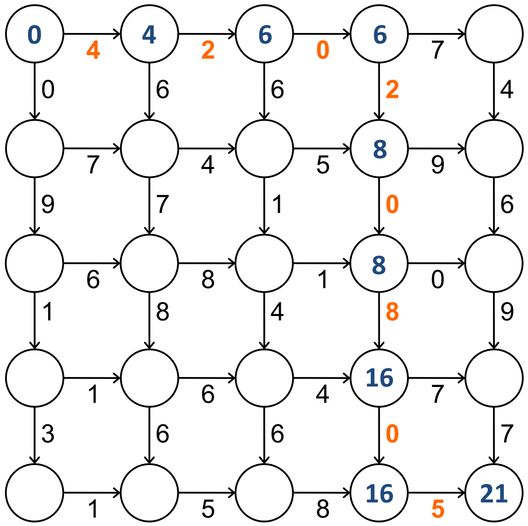
Time complexity of the "dynamic" solution

- Each recursive step takes a constant time
- Each $C(r, c)$ is evaluated at most once.
- Total time complexity is $\Theta(n_r n_c)$.
- Like Fibonacci search, the time complexity would be super exponential if $C(r, c)$ is not stored and redundantly evaluated.

Reconstructing the optimal path

- Optimal cost does not automatically produce optimal path.
- When choosing smaller-cost path between two alternatives, store the decision
- Backtrack from the destination to the source based on the stored decision

Example of backtracking the path



Implementing Manhattan tourist algorithm

Matrix615.h

```
#include <vector>

template <class T>
class Matrix615 {
public:
  std::vector< std::vector<T> > data;  // vector of vector : 2D array
  Matrix615(int nrow, int ncol, T val = 0) {
    data.resize(nrow); // make n rows
    for(int i=0; i < nrow; ++i) {
      data[i].resize(ncol,val); // make n cols with default value val
    }
  }
  int rowNums() { return (int)data.size(); }
  int colNums() { return ( data.size() == 0 ) ? 0 : (int)data[0].size(); }
};
```


Manhattan tourist problem : main()

MTP.cpp

```
#include <iostream>
#include "Matrix615.h"
int main(int argc, char** argv) {
  int nrows=5, ncols=5;
  // hw stores horizontal weights, vw stores vertical weights
  Matrix615<int> hw(nrows,ncols-1), vw(nrows-1,ncols);

  hw.data[0][0] = 4; hw.data[0][1] = 2; ...
  vw.data[0][0] = 0; vw.data[0][1] = 6; ...

  Matrix615<int> cost(nrows,ncols), move(nrows,ncols);
  // calculate the optimal cost, recording the backtracking info
  int optCost = optimalCost(hw,vw,cost,move,nrows-1,ncols-1);
  std::cout << "Optimal cost is " << optCost << std::endl;
  return 0;
}
```


Calculating optimal cost

MTP.cpp

```
// Note : must be declared before main() function
// hw, vw : horizontal and vertical input weights
// cost : stored optimal cost from (0,0) to (r,c)
// move : stored optimal decision to reach (r,c)
// r,c  : the position of interest
int optimalCost(Matrix615<int>& hw, Matrix615<int>& vw, Matrix615<int>& cost,
  Matrix615<int>& move, int r, int c) {
  if ( cost.data[r][c] == 0 ) {  // if cost is stored already, skip
    if ( ( r == 0 ) && ( c == 0 ) ) cost.data[r][c] = 0; // terminal condition
    else if ( r == 0 ) { // only horizontal move is possible
      move.data[r][c] = 0; // 0 means horitontal move to (r,c)
      cost.data[r][c] = optimalCost(hw,vw,cost,move,r,c-1) + hw.data[r][c-1];
    }
    else if ( c == 0 ) { // only vertical move is possible
      move.data[r][c] = 1; // 1 means vertical move to (r,c)
      cost.data[r][c] = optimalCost(hw,vw,cost,move,r-1,c) + vw.data[r-1][c];
    }
```

Calculating optimal cost (cont'd)

MTP.cpp

```
    else { // evaluate the cumulative cost of horizontal and vertical move
      int hcost = optimalCost(hw,vw,cost,move,r,c-1) + hw.data[r][c-1];
      int vcost = optimalCost(hw,vw,cost,move,r-1,c) + vw.data[r-1][c];
      if ( hcost > vcost ) { // when vertical move is optimal
        move.data[r][c] = 1; // store the decision
        cost.data[r][c] = vcost; // and store the optimal cost
      }
      else {
        move.data[r][c] = 0;
        cost.data[r][c] = hcost;
      }
    }
  }

  // when horizontal move is optimal
  return cost.data[r][c]; // return the optimal cost }
}
```


Dynamic programming : A smart recursion

- Dynamic programming is recursion without repetition
  1. Formulate the problem recursively
  2. Build solutions to your recurrence from the bottom up
- Dynamic programming is not about filling in tables; it's about smart recursion (Jeff Erickson)

Minimum edit distance problem

Edit distance

Minimum number of letter insertions, deletions, substitutions required to transform one word into another

An example

FOOD → MOOD → MON^D → MONED → MONEY

Edit distance is 4 in the example above

More examples of edit distance

```
F O O   D
M O N E Y

A L G O R   I   T H M
A L   T R U I S T I C
```

- Similar representation to DNA sequence alignment
- Does the above alignment provides an optimal edit distance?

Summary

Today

- Dynamic programming is a smart recursion avoiding redundancy
- Divide a problem into subproblems that can be shared
- Examples of dynamic programming
  - Fibonacci numbers
  - Manhattan tourist problem
  - Overview of edit distance problem

Next lecture

- Edit Distance
- Introduction to Hidden Markov model