# Biostatistics 602 - Statistical Inference
# Lecture 21
# Asymptotics of LRT
# Wald Test

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# Last Lecture

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- Does UMP level $\alpha$ test always exist for simple hypothesis testing?
- For composite hypothesis, which property makes it possible to construct a UMP level $\alpha$ test?
- What is a sufficient condition for an exponential family to have MLR property?
- For one-sided composite hypothesis testing, if a sufficient statistic satisfies MLR property, how can a UMP level $\alpha$ test be constructed?

# Uniformly Most Powerful Test (UMP)

## Definition

Let $\mathcal{C}$ be a class of tests between $H_0 : \theta \in \Omega$ vs $H_1 : \theta \in \Omega_0^c$. A test in $C$, with power function $\beta(\theta)$ is *uniformly most powerful (UMP) test* in class $\mathcal{C}$ if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Omega_0^c$ and every $\beta'(\theta)$, which is a power function of another test in $C$.

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Consider $\mathcal{C}$ be the set of all the level $\alpha$ test. The UMP test in this class is called a UMP level $\alpha$ test.

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# Neyman-Pearson Lemma

## Theorem 8.3.12 - Neyman-Pearson Lemma

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding the $\theta_i$ is $f(\mathbf{x}|\theta_i)$, $i = 0, 1$, using a test with rejection region $R$ that satisfies

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$$\mathbf{x} \in R \qquad \text{if } f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0) \qquad (8.3.1) \text{ and}$$
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For some $k \geq 0$ and $\alpha = \Pr(\mathbf{X} \in R|\theta_0)$, Then,

- (Sufficiency) Any test that satisfies 8.3.1 and 8.3.2 is a UMP level $\alpha$ test
- (Necessity) if there exist a test satisfying 8.3.1 and 8.3.2 with $k > 0$, then every UMP level $\alpha$ test is a size $\alpha$ test (satisfies 8.3.2), and every UMP level $\alpha$ test satisfies 8.3.1 except perhaps on a set $A$ satisfying $\Pr(\mathbf{X} \in A|\theta_0) = \Pr(\mathbf{X} \in A|\theta_1) = 0$.

# Monotone Likelihood Ratio

## Definition

A family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Omega\}$ for a univariate random variable $T$ with real-valued parameter $\theta$ have a monotone likelihood ratio if $\frac{g(t|\theta_2)}{g(t|\theta_1)}$ is an increasing (or non-decreasing) function of $t$ for every $\theta_2 > \theta_1$ on $\{t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\}$.

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Note: we may define MLR using decreasing function of $t$. But all following theorems are stated according to the definition.

# Karlin-Rabin Theorem

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Suppose $T(\mathbf{X})$ is a sufficient statistic for $\theta$ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then

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2. For testing $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$, the UMP level $\alpha$ test is given by rejecting $H_0$ if and only if $T < t_0$ where $\alpha = \Pr(T < t_0|\theta_0)$.

# Normal Example with Known Mean

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_0, \sigma^2)$ where $\sigma^2$ is unknown and $\mu_0$ is known. Find the UMP level $\alpha$ test for testing $H_0 : \sigma^2 \leq \sigma_0^2$ vs. $H_1 : \sigma^2 > \sigma_0^2$. Let $T = \sum_{i=1}^n (X_i - \mu_0)^2$ is sufficient for $\sigma^2$.

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$$\begin{aligned} \frac{X_i - \mu_0}{\sigma} &\sim \mathcal{N}(0,1) \\ \left(\frac{X_i - \mu_0}{\sigma}\right)^2 &\sim \chi_1^2 \end{aligned}$$

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f_Y(y) &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}
\end{aligned}$$

# Normal Example with Known Mean (cont'd)

$$f_T(t) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} \left(\frac{t}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}} \left|\frac{dy}{dt}\right| dt$$

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where $w(\sigma^2) = -\frac{1}{2\sigma^2}$ is an increasing function in $\sigma^2$.

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where $w(\sigma^2) = -\frac{1}{2\sigma^2}$ is an increasing function in $\sigma^2$. Therefore, $T = \sum_{i=1}^n (X_i - \mu)^2$ has the MLR property.

# Normal Example with Known Mean (cont'd)

By Karlin-Rabin Theorem, UMP level $\alpha$ rejects $s$ $H_0$ if and only if $T > t_0$ where $t_0$ is chosen such that $\alpha = \Pr(T > t_0|\sigma_0^2)$.

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Note that $\frac{T}{\sigma^2} \sim \chi_n^2$

$$\Pr(T > t_0|\sigma_0^2) \quad = \quad \Pr\left(\frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2}\middle|\,\sigma_0^2\right)$$

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\Pr\left(\chi_n^2 > \frac{t_0}{\sigma_0^2}\right) &= \alpha \\
\frac{t_0}{\sigma_0^2} &= \chi_{n,\alpha}^2
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\Pr\left(\chi^2_n > \frac{t_0}{\sigma_0^2}\right) &= \alpha \\
\frac{t_0}{\sigma_0^2} &= \chi^2_{n,\alpha} \\
t_0 &= \sigma_0^2 \chi^2_{n,\alpha}
\end{aligned}$$

where $\chi^2_{n,\alpha}$ satisfies $\int_{\chi^2_{n,\alpha}}^{\infty} f_{\chi^2_n}(x)dx = \alpha$.

# Remarks

- For many problems, UMP level $\alpha$ test does not exist (Example 8.3.19).

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- For many problems, UMP level $\alpha$ test does not exist (Example 8.3.19).
- In such cases, we can restrict our search among a subset of tests, for example, all unbiased tests.

# Distribution of LRT

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$$\sup_{\theta \in \Omega_0} \Pr(\lambda(\mathbf{x}) \leq c) \leq \alpha$$

Usually, it is difficult to derive the distribution of $\lambda(\mathbf{x})$ and to solve the equation of $c$.

# Asymptotics of LRT

## Theorem 10.3.1

Consider testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose $X_1, \cdots, X_n$ are iid samples from $f(x|\theta)$, and $\hat{\theta}$ is the MLE of $\theta$, and $f(x|\theta)$ satisfies certain "regularity conditions" (e.g. see misc 10.6.2), then under $H_0$:

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$$-2 \log \lambda(\mathbf{x}) \xrightarrow{\mathrm{d}} \chi^2_1$$

as $n \to \infty$.

# Proof

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Omega_0} L(\theta|\mathbf{x})}{\sup_{\theta \in \Omega} L(\theta|\mathbf{x})} = \frac{L(\theta_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}$$

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$$-2\lambda(\mathbf{x}) = -2\log\left(\frac{L(\theta_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}\right)$$

$$= -2\log L(\theta_0|\mathbf{x}) + 2\log L(\hat{\theta}|\mathbf{x})$$

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$$\begin{aligned}
-2\lambda(\mathbf{x}) &= -2\log\left(\frac{L(\theta_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}\right) \\
&= -2\log L(\theta_0|\mathbf{x}) + 2\log L(\hat{\theta}|\mathbf{x}) \\
&= -2l(\theta_0|\mathbf{x}) + 2l(\hat{\theta}|\mathbf{x})
\end{aligned}$$

# Proof (cont'd)

Expanding $l(\theta|\mathbf{x})$ around $\hat{\theta}$,

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$$l(\theta|\mathbf{x}) = l(\hat{\theta}|\mathbf{x}) + l'(\hat{\theta}|\mathbf{x})(\theta - \hat{\theta}) + l''(\hat{\theta}|\mathbf{x})\frac{(\theta - \hat{\theta})^2}{2} + \cdots$$

# Proof (cont'd)

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$$\begin{aligned} l(\theta|\mathbf{x}) &= l(\hat{\theta}|\mathbf{x}) + l'(\hat{\theta}|\mathbf{x})(\theta - \hat{\theta}) + l''(\hat{\theta}|\mathbf{x})\frac{(\theta - \hat{\theta})^2}{2} + \cdots \\ l'(\hat{\theta}|\mathbf{x}) &= 0 \qquad \text{(assuming regularity condition)} \end{aligned}$$

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l(\theta_0|\mathbf{x}) &\approx l(\hat{\theta}|\mathbf{x}) + l''(\hat{\theta}|\mathbf{x})\frac{(\theta_0 - \hat{\theta})^2}{2}
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&\approx -(\theta_0 - \hat{\theta})^2 l''(\hat{\theta}|\mathbf{x})
\end{aligned}$$

# Proof (cont'd)

Because $\hat{\theta}$ is MLE, under $H_0$,

$$\hat{\theta} \quad \sim \quad \mathcal{AN}\left(\theta_0, \frac{1}{I_n(\theta_0)}\right)$$

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Therefore,

$$-2\log\lambda(\mathbf{x}) \approx -(\theta_0 - \hat{\theta})^2 l''(\hat{\theta}|\mathbf{x})$$

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$$\begin{aligned} -2\log\lambda(\mathbf{x}) &\approx -(\theta_0 - \hat{\theta})^2 l''(\hat{\theta}|\mathbf{x}) \\ &= (\hat{\theta} - \theta_0)^2 I_n(\theta_0) \frac{-\frac{1}{n} l''(\hat{\theta}|\mathbf{x})}{\frac{1}{n} I_n(\theta_0)} \end{aligned}$$

# Proof (cont'd)

$$-\frac{1}{n}l''(\hat{\theta}|\mathbf{x}) = -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta^2}f(x_i|\theta)\bigg|_{\theta=\hat{\theta}}$$

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By Slutsky's Theorem, under $H_0$

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# Example

$X_i \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda)$. Consider testing $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda \neq \lambda_0$.

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$$\lambda(\mathbf{x}) = \frac{L(\lambda_0|\mathbf{x})}{\sup_\lambda L(\lambda|\mathbf{x})}$$

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$$\lambda(\mathbf{x}) = \frac{\prod_{i=1}^n \frac{e^{-\lambda_0}\lambda_0^{x_i}}{x_i!}}{\prod_{i=1}^n \frac{e^{-\overline{x}}\overline{x}^{x_i}}{x_i!}}$$

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&= \frac{e^{-n\lambda_0}\lambda_0^{\sum x_i}}{e^{-n\overline{x}}\overline{x}^{\sum x_i}} \\
&= e^{-n(\lambda_0-\overline{x})}\left(\frac{\lambda_0}{\overline{x}}\right)^{\sum x_i}
\end{aligned}$$

# Example (cont'd)

LRT is to reject $H_0$ when $\lambda(\mathbf{x}) \leq c$

$$\alpha = \Pr(\lambda(\mathbf{X}) \leq c | \lambda_0)$$

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$$\begin{aligned} \alpha &= \Pr(\lambda(\mathbf{X}) \leq c|\lambda_0) \\ -2\log\lambda(\mathbf{X}) &= -2\left[-n(\lambda_0 - \overline{X}) + \sum X_i(\log\lambda_0 - \log\overline{X})\right] \end{aligned}$$

# Example (cont'd)

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under $H_0$, (by Theorem 10.3.1).

# Example (cont'd)

Therefore, asymptotic size $\alpha$ test is

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$$\begin{aligned} \Pr(\lambda(\mathbf{X}) \leq c|\lambda_0) &= \alpha \\ \Pr(-2\log\lambda(\mathbf{X}) \leq c^*|\lambda_0) &= \alpha \end{aligned}$$

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Therefore, asymptotic size $\alpha$ test is

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\Pr(\chi^2_1 \geq c^*) &\approx \alpha
\end{aligned}$$

# Example (cont'd)

Therefore, asymptotic size $\alpha$ test is

$$\begin{aligned}
\Pr(\lambda(\mathbf{X}) \leq c|\lambda_0) &= \alpha \\
\Pr(-2\log\lambda(\mathbf{X}) \leq c^*|\lambda_0) &= \alpha \\
\Pr(\chi^2_1 \geq c^*) &\approx \alpha \\
c^* &= \chi^2_{1,\alpha}
\end{aligned}$$

rejects $H_0$ if and only if $-2\log\lambda(\mathbf{x}) \geq \chi^2_{1,\alpha}$

# Wald Test

Wald test relates point estimator of $\theta$ to hypothesis testing about $\theta$.

## Definition

Syppose $W_n$ is an estimator of $\theta$ and $W_n \sim \mathcal{AN}(\theta, \sigma_W^2)$. Then Wald test statistic is defined as

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where $\theta_0$ is the value of $\theta$ under $H_0$ and $S_n$ is a consistent estimator of $\sigma_W$

# Examples of Wald Test

## Two-sided Wald Test

$H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$, then Wald asymptotic level $\alpha$ test is to reject $H_0$ if and only if

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## One-sided Wald Test

$H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$, then Wald asymptotic level $\alpha$ test is to reject $H_0$ if and only if

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# Remarks

- Different estimators of $\theta$ leads to different testing procedures.

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- One choice of $W_n$ is MLE and we may choose $S_n = \frac{1}{I_n(W_n)}$ or $\frac{1}{I_n(\hat{\theta})}$ (observed information number) when $\sigma_W^2 = \frac{1}{I_n(\theta)}$.

# Example of Wald Test

Suppose $X_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$, and consider testing $H_0 : p = p_0$ vs $H_1 : p \neq p_0$.

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$$\overline{X} \sim \mathcal{AN}\left(p, \frac{p(1-p)}{n}\right)$$

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# Example of Wald Test

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by the Central Limit Theorem. The Wald test statistic is

$$Z_n = \frac{\overline{X} - p_0}{S_n}$$

where $S_n$ is a consistent estimator of $\sqrt{\frac{p(1-p)}{n}}$, whose MLE is

$$S_n = \sqrt{\frac{\overline{X}(1-\overline{X})}{n}}$$

by the invariance property of MLE.

# Example of Wald Test (cont'd)

Therefore, $S_n$ is consistent for $\sqrt{\frac{p(1-p)}{n}}$. The Wald statistic is

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## Example of Wald Test (cont'd)

Therefore, $S_n$ is consistent for $\sqrt{\frac{p(1-p)}{n}}$. The Wald statistic is

$$Z_n = \frac{\overline{X} - p_0}{\sqrt{\overline{X}(1-\overline{X})/n}}$$

An asymptotic level $\alpha$ Wald test rejects $H_0$ if and only if

$$\left| \frac{\overline{X} - p_0}{\sqrt{\overline{X}(1-\overline{X})/n}} \right| > z_{\alpha/2}$$

# Summary

## Today

- Asymptotics of LRT
- Wald Test

## Next Week

- p-Values