Biostatistics 602 - Statistical Inference
Lecture 20
Uniformly Most Powerful Test

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Last Lecture

- What are the typical steps for constructing a likelihood ratio test?
- Is LRT statistic based on sufficient statistic identical to the LRT based on the full data?
- When multiple parameters need to be estimated, what is the difference in constructing LRT?
- What is unbiased test?

LRT based on sufficient statistics

Theorem 8.2.4

If $T(\mathbf{X})$ is a sufficient statistic for θ, $\lambda^*(t)$ is the LRT statistic based on T, and $\lambda(\mathbf{x})$ is the LRT statistic based on $\mathbf{x}$ then

$$\lambda^*[T(\mathbf{x})] = \lambda(\mathbf{x})$$

for every $\mathbf{x}$ in the sample space.

Unbiased Test

Definition

If a test always satisfies

$$\Pr(\text{reject } H_0 \text{ when } H_0 \text{ is false }) \geq \Pr(\text{reject } H_0 \text{ when } H_0 \text{ is true })$$

Then the test is said to be unbiased

Alternative Definition

Recall that $\beta(\theta) = \Pr(\text{reject } H_0)$. A test is unbiased if

$$\beta(\theta') \geq \beta(\theta)$$

for every $\theta' \in \Omega_0^c$ and $\theta \in \Omega_0$.

Example

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known, testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.

LRT test rejects H_0 if $\frac{\bar{x}-\theta_0}{\sigma/\sqrt{n}} > c$.

$$\begin{aligned}\beta(\theta) &= \Pr\left(\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} > c\right) \\ &= \Pr\left(\frac{\overline{X}-\theta+\theta-\theta_0}{\sigma/\sqrt{n}} > c\right) \\ &= \Pr\left(\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} + \frac{\theta-\theta_0}{\sigma/\sqrt{n}} > c\right) \\ &= \Pr\left(\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} > c - \frac{\theta-\theta_0}{\sigma/\sqrt{n}}\right)\end{aligned}$$

Note that $X_i \sim \mathcal{N}(\theta, \sigma^2)$, $\overline{X} \sim \mathcal{N}(\theta, \sigma^2/n)$, and $\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$.

Example (cont'd)

Therefore, for $Z \sim \mathcal{N}(0,1)$

$$\beta(\theta) = \Pr\left(Z > c + \frac{\theta_0-\theta}{\sigma/\sqrt{n}}\right)$$

Because the power function is increasing function of θ,

$$\beta(\theta') \geq \beta(\theta)$$

always holds when $\theta \leq \theta_0 < \theta'$. Therefore the LRTs are unbiased.

Uniformly Most Powerful Test (UMP)

Definition

Let $\mathcal{C}$ be a class of tests between $H_0 : \theta \in \Omega$ vs $H_1 : \theta \in \Omega_0^c$. A test in C, with power function $\beta(\theta)$ is *uniformly most powerful (UMP) test* in class $\mathcal{C}$ if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Omega_0^c$ and every $\beta'(\theta)$, which is a power function of another test in C.

UMP level α test

Consider $\mathcal{C}$ be the set of all the level α test. The UMP test in this class is called a UMP level α test.

UMP level α test has the smallest type II error probability for any $\theta \in \Omega_0^c$ in this class.

- A UMP test is "uniform" in the sense that it is most powerful for every $\theta \in \Omega_0^c$.
- For simple hypothesis such as $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$, UMP level α test always exists.

Neyman-Pearson Lemma

Theorem 8.3.12 - Neyman-Pearson Lemma

Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where the pdf or pmf corresponding the θ_i is $f(\mathbf{x}|\theta_i)$, $i = 0, 1$, using a test with rejection region R that satisfies

$$\mathbf{x} \in R \quad \text{if } f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0) \quad (8.3.1) \text{ and}$$
$$\mathbf{x} \in R^c \quad \text{if } f(\mathbf{x}|\theta_1) < kf(\mathbf{x}|\theta_0) \quad (8.3.2)$$

For some $k \geq 0$ and $\alpha = \Pr(\mathbf{X} \in R|\theta_0)$, Then,

- (Sufficiency) Any test that satisfies 8.3.1 and 8.3.2 is a UMP level α test
- (Necessity) if there exist a test satisfying 8.3.1 and 8.3.2 with $k > 0$, then every UMP level α test is a size α test (satisfies 8.3.2), and every UMP level α test satisfies 8.3.1 except perhaps on a set A satisfying $\Pr(\mathbf{X} \in A|\theta_0) = \Pr(\mathbf{X} \in A|\theta_1) = 0$.

Example of Neyman-Pearson Lemma

Let $X \in \text{Binomial}(2, \theta)$, and consider testing $H_0 : \theta = \theta_0 = 1/2$ vs. $H_1 : \theta = \theta_1 = 3/4$.
Calculating the ratios of the pmfs given,

$$\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{1}{4}, \qquad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{3}{4}, \qquad \frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{9}{4}$$

- Suppose that $k < 1/4$, then the rejection region $R = \{0, 1, 2\}$, and UMP level α test always rejects H_0. Therefore $\alpha = \Pr(\text{reject } H_0|\theta = \theta_0 = 1/2) = 1$.
- Suppose that $1/4 < k < 3/4$, then $R = \{1, 2\}$, and UMP level α test rejects H_0 if $x = 1$ or $x = 2$.

$$\alpha = \Pr(\text{reject}|\theta = 1/2) = \Pr(x = 1|\theta = 1/2) + \Pr(x = 2|\theta = 1/2) = \frac{3}{4}$$

Example of Neyman-Pearson Lemma (cont'd)

- Suppose that $3/4 < k < 9/4$, then UMP level α test rejects H_0 if $x = 2$

$$\alpha = \Pr(\text{reject}|\theta = 1/2) = \Pr(x = 2|\theta = 1/2) = \frac{1}{4}$$

- If $k > 9/4$ the UMP level α test always not reject H_0, and $\alpha = 0$

Example - Normal Distribution

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.

$$f(\mathbf{x}|\theta) = \prod_{i=1}^n \left[\frac{1}{2\pi\sigma^2} \exp\left\{-\frac{(x_i - \theta)^2}{2\sigma^2}\right\}\right]$$

$$\begin{aligned}
\frac{f(\mathbf{x}|\theta_1)}{f(\mathbf{x}|\theta_0)} &= \frac{\exp\left\{-\frac{\sum_{i=1}^n (x_i - \theta_1)^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{\sum_{i=1}^n (x_i - \theta_0)^2}{2\sigma^2}\right\}} \\
&= \exp\left[-\frac{\sum_{i=1}^n (x_i - \theta_1)^2}{2\sigma^2} + \frac{\sum_{i=1}^n (x_i - \theta_0)^2}{2\sigma^2}\right] \\
&= \exp\left[\frac{\sum_{i=1}^n (x_i - \theta_0)^2 - \sum_{i=1}^n (x_i - \theta_1)^2}{2\sigma^2}\right] \\
&= \exp\left[\frac{n(\theta_0^2 - \theta_1)^2 + 2\sum_{i=1}^n x_i(\theta_1 - \theta_0)}{2\sigma^2}\right]
\end{aligned}$$

Example (cont'd)

UMP level α test rejects if

$$\exp\left[\frac{n(\theta_0^2-\theta_1)^2 + 2\sum_{i=1}^n x_i(\theta_1-\theta_0)}{2\sigma^2}\right] > k$$

$$\iff \frac{n(\theta_0^2-\theta_1)^2 + 2\sum_{i=1}^n x_i(\theta_1-\theta_0)}{2\sigma^2} > \log k$$

$$\iff \sum_{i=1}^n x_i > k^*$$

$$\alpha = \Pr\left(\sum_{i=1}^n X_i > k^* | \theta_0\right)$$

Example (cont'd)

Under H_0,

$$\begin{aligned} X_i &\sim \mathcal{N}(\theta_0, \sigma^2) \\ \overline{X} &\sim \mathcal{N}(\theta_0, \sigma^2/n) \\ \frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} &\sim \mathcal{N}(0,1) \end{aligned}$$

$$\begin{aligned} \alpha &= \Pr\left(\sum_{i=1}^n X_i > k^* | \theta_0\right) \\ &= \Pr\left(Z > \frac{k^*/n - \theta_0}{\sigma/\sqrt{n}}\right) \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$.

Example (cont'd)

$$\begin{aligned} \frac{k^*/n - \theta_0}{\sigma/\sqrt{n}} &= z_\alpha \\ k^* &= n\left(\theta_0 + z_\alpha \frac{\sigma}{\sqrt{n}}\right) \end{aligned}$$

Thus, the UMP level α test reject if $\sum X_i > k^*$, or equivalently, reject H_0 if $\overline{X} > k^*/n = \theta_0 + z_\alpha \sigma/\sqrt{n}$

Neyman-Pearson Lemma on Sufficient Statistics

Corollary 8.3.13

Consider $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Suppose $T(\mathbf{X})$ is a sufficient statistic for θ and $g(t|\theta_i)$ is the pdf or pmf of T. Corresponding $\theta_i, i \in \{0,1\}$. Then any test based on T with rejection region S is a UMP level α test if it satisfies

$$\begin{aligned} t \in S & \quad \text{if } g(t|\theta_1) > k \cdot g(t|\theta_0) \text{ and} \\ t \in S^c & \quad \text{if } g(t|\theta_1) < k \cdot g(t|\theta_0) \end{aligned}$$

For some $k > 0$ and $\alpha = \Pr(T \in S|\theta_0)$

Proof

The rejection region in the sample space is

$$R = \{\mathbf{x} : T(\mathbf{x}) = t \in S\} = \{\mathbf{x} : g(T(\mathbf{x})|\theta_1) > kg(T(\mathbf{x})|\theta_0)\}$$

By Factorization Theorem:

$$f(\mathbf{x}|\theta_i) = h(\mathbf{x})g(T(\mathbf{x})|\theta_i)$$
$$R = \{\mathbf{x} : g(T(\mathbf{x})|\theta_1)h(x) > kg(T(\mathbf{x})|\theta_0)h(x)\} = \{\mathbf{x} : f(\mathbf{x}|\theta_1) > kf(\mathbf{x}|\theta_0)\}$$

By Neyman-Pearson Lemma, this test is the UMP level α test, and

$$\alpha = \Pr(\mathbf{X} \in R) = \Pr(T(\mathbf{X}) \in S|\theta_0)$$

Revisiting the Example of Normal Distribution

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_1 > \theta_0$.
$T = \overline{X}$ is a sufficient statistic for θ, where $T \sim \mathcal{N}(\theta, \sigma^2/n)$.

$$g(t|\theta_i) = \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{(t-\theta_i)^2}{2\sigma^2/n}\right\}$$

$$\frac{g(t|\theta_1)}{g(t|\theta_0)} = \frac{\exp\left\{-\frac{(t-\theta_1)^2}{2\sigma^2/n}\right\}}{\exp\left\{-\frac{(t-\theta_0)^2}{2\sigma^2/n}\right\}} = \exp\left\{-\frac{1}{2\sigma^2/n}\left[(t-\theta_1)^2 - (t-\theta_0)^2\right]\right\} = \exp\left\{-\frac{1}{2\sigma^2/n}\left[\theta_1^2 - \theta_0^2 - 2t(\theta_1-\theta_0)\right]\right\}$$

Revisiting the Example (cont'd)

UMP level α test reject if

$$\exp\left\{-\frac{1}{2\sigma^2/n}\left[\theta_1^2 - \theta_0^2 - 2t(\theta_1-\theta_0)\right]\right\} > k$$
$$\iff \frac{1}{2\sigma^2/n}\left[-(\theta_1^2 - \theta_0^2) + 2t(\theta_1-\theta_0)\right] > \log k$$
$$\iff \overline{X} = t > k^*$$

Revisiting the Example (cont'd)

Under H_0, $\overline{X} \sim \mathcal{N}(\theta_0, \sigma^2/n)$. k^* satisfies

$$\Pr(\text{reject } H_0|\theta_0) = \alpha$$
$$\alpha = \Pr(\overline{X} > k^*|\theta_0) = \Pr\left(\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} > \frac{k^*-\theta_0}{\sigma/\sqrt{n}}\right) = \Pr\left(Z > \frac{k^*-\theta_0}{\sigma/\sqrt{n}}\right)$$
$$\frac{k^*-\theta_0}{\sigma/\sqrt{n}} = z_\alpha$$
$$k^* = \theta_0 + z_\alpha \frac{\sigma}{n}$$

Monotone Likelihood Ratio

Definition

A family of pdfs or pmfs $\{g(t|\theta) : \theta \in \Omega\}$ for a univariate random variable T with real-valued parameter θ have a monotone likelihood ratio if $\frac{g(t|\theta_2)}{g(t|\theta_1)}$ is an increasing (or non-decreasing) function of t for every $\theta_2 > \theta_1$ on $\{t : g(t|\theta_1) > 0 \text{ or } g(t|\theta_2) > 0\}$.

Note: we may define MLR using decreasing function of t. But all following theorems are stated according to the definition.

Example of Monotone Likelihood Ratio

- Normal, Poisson, Binomial have the MLR Property (Exercise 8.25)
- If T is from an exponential family with the pdf or pmf

$$g(t|\theta) = h(t)c(\theta)\exp[w(\theta) \cdot t]$$

Then T has an MLR if $w(\theta)$ is a non-decreasing function of θ.

Proof

Suppose that $\theta_2 > \theta_1$.

$$\begin{aligned} \frac{g(t|\theta_2)}{g(t|\theta_1)} &= \frac{h(t)c(\theta_2)\exp[w(\theta_2)t]}{h(t)c(\theta_1)\exp[w(\theta_1)t]} \\ &= \frac{c(\theta_2)}{c(\theta_1)}\exp[\{w(\theta_2) - w(\theta_1)\}t] \end{aligned}$$

If $w(\theta)$ is a non-decreasing function of θ, then $w(\theta_2) - w(\theta_1) \geq 0$ and $\exp[\{w(\theta_2) - w(\theta_1)\}t]$ is an increasing function of t. Therefore, $\frac{g(t|\theta_2)}{g(t|\theta_1)}$ is a non-decreasing function of t, and T has MLR if $w(\theta)$ is a non-decreasing function of θ.

Karlin-Rabin Theorem

Theorem 8.1.17

Suppose $T(\mathbf{X})$ is a sufficient statistic for θ and the family $\{g(t|\theta) : \theta \in \Omega\}$ is an MLR family. Then

1. For testing $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$, the UMP level α test is given by rejecting H_0 is and only if $T > t_0$ where $\alpha = \Pr(T > t_0|\theta_0)$.
2. For testing $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$, the UMP level α test is given by rejecting H_0 if and only if $T < t_0$ where $\alpha = \Pr(T < t_0|\theta_0)$.

Example Application of Karlin-Rabin Theorem

Let $X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known, Find the UMP level α test for $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$.
$T(\mathbf{X}) = \overline{X}$ is a sufficient statistic for θ, and $T \sim \mathcal{N}(\theta, \sigma^2/n)$.

$$\begin{aligned} g(t|\theta) &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{(t-\theta)^2}{2\sigma^2/n}\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{t^2+\theta^2-2t\theta}{2\sigma^2/n}\right\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{-\frac{t^2}{2\sigma^2/n}\right\} \exp\left\{-\frac{\theta^2}{2\sigma^2/n}\right\} \exp\left\{\frac{t\theta}{\sigma^2/n}\right\} \\ &= h(t)c(\theta)\exp[w(\theta)t] \end{aligned}$$

where $w(\theta) = \frac{\theta}{\sigma^2/n}$ is an increasing function in θ. Therefore T is MLR property.

Finding a UMP level α test

By Karlin-Rabin, UMP level α test rejects H_0 iff. $T > t_0$ where

$$\begin{aligned} \alpha &= \Pr(T > t_0|\theta_0) \\ &= \Pr\left(\frac{T-\theta_0}{\sigma/\sqrt{n}} > \frac{t_0-\theta_0}{\sigma/\sqrt{n}} \middle| \theta_0\right) \\ &= \Pr\left(Z > \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right) \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$.

$$\begin{aligned} \frac{t_0-\theta_0}{\sigma/\sqrt{n}} &= z_\alpha \\ \Rightarrow t_0 &= \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha \end{aligned}$$

UMP level α test rejects H_0 if $T = \overline{X} > \theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha$.

Testing $H_0 : \theta \geq \theta_0$ vs. $H_1 : \theta < \theta_0$

UMP level α test rejects H_0 if $T < t_0$ where

$$\begin{aligned} \alpha &= \Pr(T < t_0|\theta_0) = \Pr\left(\frac{T-\theta_0}{\sigma/\sqrt{n}} < \frac{t_0-\theta_0}{\sigma/\sqrt{n}} \middle| \theta_0\right) \\ &= \Pr\left(Z < \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right) \\ 1-\alpha &= \Pr\left(Z \geq \frac{t_0-\theta_0}{\sigma/\sqrt{n}}\right) \\ \frac{t_0-\theta_0}{\sigma/\sqrt{n}} &= z_{1-\alpha} \\ t_0 &= \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} = \theta_0 - \frac{\sigma}{\sqrt{n}} z_\alpha \end{aligned}$$

Therefore, the test rejects H_0 if $T < t_0 = \theta - \frac{\sigma}{\sqrt{n}} z_\alpha$

Normal Example with Known Mean

$X_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_0, \sigma^2)$ where σ^2 is unknown and μ_0 is known. Find the UMP level α test for testing $H_0 : \sigma^2 \leq \sigma_0^2$ vs. $H_1 : \sigma^2 > \sigma_0^2$. Let $T = \sum_{i=1}^n (X_i - \mu_0)^2$ is sufficient for σ^2. To check whether T has MLR property, we need to find $g(t|\sigma^2)$.

$$\begin{aligned} \frac{X_i - \mu_0}{\sigma} &\sim \mathcal{N}(0,1) \\ \left(\frac{X_i-\mu_0}{\sigma}\right)^2 &\sim \chi_1^2 \\ Y &= T/\sigma^2 = \sum_{i=1}^n \left(\frac{X_i-\mu_0}{\sigma}\right)^2 \sim \chi_n^2 \\ f_Y(y) &= \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{n/2}} y^{\frac{n}{2}-1} e^{-\frac{y}{2}} \end{aligned}$$

Normal Example with Known Mean (cont'd)

$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma\left(\frac{n}{2}\right)2^{n/2}}\left(\frac{t}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}}\left|\frac{dy}{dt}\right| \\ &= \frac{1}{\Gamma\left(\frac{n}{2}\right)2^{n/2}}\left(\frac{t}{\sigma^2}\right)^{\frac{n}{2}-1} e^{-\frac{t}{2\sigma^2}}\frac{1}{\sigma^2} \\ &= \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)2^{n/2}}\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{t}{2\sigma^2}} \\ &= h(t)c(\sigma^2)\exp[w(\sigma^2)t] \end{aligned}$$

where $w(\sigma^2) = -\frac{1}{2\sigma^2}$ is an increasing function in σ^2. Therefore, $T = \sum_{i=1}^n (X_i - \mu_0)^2$ has the MLR property.

Normal Example with Known Mean (cont'd)

By Karlin-Rabin Theorem, UMP level α rejects s H_0 if and only if $T > t_0$ where t_0 is chosen such that $\alpha = \Pr(T > t_0|\sigma_0^2)$.
Note that $\frac{T}{\sigma^2} \sim \chi^2_n$

$$\begin{aligned} \Pr(T > t_0|\sigma_0^2) &= \Pr\left(\frac{T}{\sigma_0^2} > \frac{t_0}{\sigma_0^2}\middle|\sigma_0^2\right) \\ \frac{T}{\sigma_0^2} &\sim \chi^2_n \\ \Pr\left(\chi^2_n > \frac{t_0}{\sigma_0^2}\right) &= \alpha \\ \frac{t_0}{\sigma_0^2} &= \chi^2_{n,\alpha} \\ t_0 &= \sigma_0^2\chi^2_{n,\alpha} \end{aligned}$$

where $\chi^2_{n,\alpha}$ satisfies $\int_{\chi^2_{n,\alpha}}^{\infty} f_{\chi^2_n}(x)dx = \alpha$.

Remarks

- For many problems, UMP level α test does not exist (Example 8.3.19).
- In such cases, we can restrict our search among a subset of tests, for example, all unbiased tests.

Summary

Today

- Uniformly Most Powerful Test
- Neyman-Pearson Lemma
- Monotone Likelihood Ratio
- Karlin-Rabin Theorem

Next Lecture

- Asymptotics of LRT
- Wald Test