Biostatistics 602 - Statistical Inference
Lecture 23
Interval Estimation

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Last Lecture

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- What is p-value?

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- Is Fisher's exact p-value uniformly distributed under null hypothesis?

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A *p-value* $p(\mathbf{X})$ is a test statistic satisfying $0 \leq p(\mathbf{x}) \leq 1$ for every sample point $\mathbf{x}$. Small values of $p(\mathbf{X})$ given evidence that H_1 is true. A *p-value* is *valid* if, for every $\theta \in \Omega_0$ and every $0 \leq \alpha \leq 1$,

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$$\Pr(p(\mathbf{X}) \leq \alpha | \theta) \leq \alpha$$

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Theorem 8.3.27.

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Then $p(\mathbf{X})$ is a valid p-value.

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Suppose $S(\mathbf{X})$ is a sufficient statistic for the model $\{f(\mathbf{x}|\theta) : \theta \in \Omega_0\}$. (not necessarily including alternative hypothesis).

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$$p(\mathbf{x}) \quad = \quad \Pr(\, W(\mathbf{X}) \geq \, W(\mathbf{x})|S = S(\mathbf{x}))$$

If we consider only the conditional distribution, by Theorem 8.3.27, this is a valid p-value, meaning that

$$\Pr(p(\mathbf{X}) \leq \alpha | S = s) \leq \alpha$$

Example - Fisher's Exact Test

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Let X_1 and X_2 be independent observations with $X_1 \sim \text{Binomial}(n_1, p_1)$, and $X_2 \sim \text{Binomial}(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$. Find a valid p-value function.

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Therefore $S = X_1 + X_2$ is a sufficient statistic under H_0.

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Given the value of $S = s$, it is reasonable to use X_1 as a test statistic and reject H_0 in favor of H_1 for large values of X_1,because large values of X_1 correspond to small values of $X_2 = s - X_1$.

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$$f(X_1 = x_1|s) = \frac{\binom{n_1}{x_1}\binom{n_2}{s-x_1}}{\binom{n_1+n_2}{s}}$$

Thus, the p-value conditional on the sufficient statistic $s = x_1 + x_2$ is

$$p(x_1, x_2) = \sum_{j=x_1}^{\min(n_1,s)} f(j|s)$$

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&= \Pr(-\sqrt{n} \leq Z \leq \sqrt{n}) \xrightarrow{\text{P}} 1
\end{aligned}$$

as $n \to \infty$, where $Z \sim \mathcal{N}(0, 1)$.

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where $\mathbf{X}$ are random samples from $f_{\mathbf{X}}(\mathbf{x}|\theta)$. In other words, it is the average length of the interval estimator.

How to construct confidence interval?

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There is a one-to-one correspondence between tests and confidence intervals (or confidence sets).

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Acceptance region is $\left\{ \mathbf{x} : \theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{x} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}$

Example (cont'd)

As this is size α test, the probability of accepting H_0 is $1 - \alpha$.

$$1 - \alpha = \Pr\left(\theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{X} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right)$$

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Since θ_0 is arbitrary,

$$1-\alpha = \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right)$$

Example (cont'd)

As this is size α test, the probability of accepting H_0 is $1-\alpha$.

$$\begin{aligned} 1-\alpha &= \Pr\left(\theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{X} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) \\ &= \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta_0 \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) \end{aligned}$$

Since θ_0 is arbitrary,

$$1-\alpha = \Pr\left(\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \theta \leq \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right)$$

Therefore, $[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}]$ is $(1-\alpha)$ confidence interval (CI).

Confidence intervals and level α test

Theorem 9.2.2

1. For each $\theta_0 \in \Omega$, let $A(\theta_0)$ be the acceptance region of a level α test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ Define a set $C(\mathbf{X}) = \{\theta : \mathbf{x} \in A(\theta)\}$, then the random set $C(\mathbf{X})$ is a $1 - \alpha$ confidence set.

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2. Conversely, if $C(\mathbf{X})$ is a $(1 - \alpha)$ confidence set for θ, for any θ_0, define the acceptance region of a test for the hypothesis $H_0 : \theta = \theta_0$ by $A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$. Then the test has level α.

Example

For $X_i \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$, the acceptance region $A(\theta_0)$ is a subset of the sample space

$$A(\theta_0) = \left\{ \mathbf{x} : \theta_0 - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \leq \overline{X} \leq \theta_0 + \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}$$

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The confidence set $C(\mathbf{X})$ is a subset of the parameter space

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Example

Problem

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1. Find $1 - \alpha$ two-sided CI for μ
2. Find $1 - \alpha$ upper bound for μ

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The CI is $(-\infty, U(\mathbf{X})]$. We need to invert a testing procedure for $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$.

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$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0, \hat{\sigma}_0^2|\mathbf{x})}{L(\hat{\mu}, \hat{\sigma}^2|\mathbf{x})}$$

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Example - upper bounded CI - Solution (cont'd)

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Example - upper bounded CI - Solution (cont'd)

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$$\begin{cases} \hat{\mu} = \overline{X} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n} \quad \text{if } \overline{X} \leq \mu_0 \\ \end{cases}$$

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$$\lambda(\mathbf{x}) = \begin{cases} 1 & \text{if } \overline{X} > \mu_0 \\ \frac{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{2\hat{\sigma}_0^2}\right\}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{\sum_{i=1}^n (X_i - \overline{X})^2}{2\hat{\sigma}_0^2}\right\}} & \text{if } \overline{X} \leq \mu_0 \end{cases}$$

Example - upper bounded CI - Solution (cont'd)

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Example - upper bounded CI - Solution (cont'd)

For $0 < c < 1$, LRT test rejects H_0 if $\overline{X} \leq \mu_0$ and

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Example - upper bounded CI - Solution (cont'd)

For $0 < c < 1$, LRT test rejects H_0 if $\overline{X} \leq \mu_0$ and

$$\begin{aligned}
\left(\frac{\frac{n-1}{n} s^2_{\mathbf{X}}}{\frac{n-1}{n} s^2_{\mathbf{X}} + (\overline{X} - \mu_0)^2} \right)^{\frac{n}{2}} &< c \\
\left(\frac{\frac{n-1}{n}}{\frac{n-1}{n} + \frac{(\overline{X}-\mu_0)^2}{s^2_{\mathbf{x}}}} \right)^{\frac{n}{2}} &< c
\end{aligned}$$

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$$\frac{(\overline{X} - \mu_0)^2}{s_{\mathbf{X}}^2} > c^*$$

$$\frac{\mu_0 - \overline{X}}{s_{\mathbf{X}} / \sqrt{n}} > c^{**}$$

Example - upper bounded CI - Solution (cont'd)

c^{**} is chosen to satisfy

$$\alpha = \Pr(\text{reject } H_0 | \mu_0)$$

Example - upper bounded CI - Solution (cont'd)

c^{**} is chosen to satisfy

$$\begin{aligned}\alpha &= \Pr(\text{reject } H_0|\mu_0) \\ &= \Pr\left(\frac{\mu_0 - \overline{X}}{s_{\mathbf{X}}/\sqrt{n}} > c^{**}\right)\end{aligned}$$

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&= \Pr(T_{n-1} < -c^{**})
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Therefore, LRT level α test reject H_0 if

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Therefore, LRT level α test reject H_0 if

$$\frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} < -t_{n-1,\alpha}$$

Example - upper bounded CI - Solution (cont'd)

Acceptance region is

$$A(\mu_0) = \left\{ \mathbf{x} : \frac{\overline{X} - \mu_0}{s_{\mathbf{X}}/\sqrt{n}} \geq -t_{n-1,\alpha} \right\}$$

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$$C(\mathbf{X}) \quad = \quad \{\mu : \mathbf{X} \in A(\mu)\}$$

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Inverting the above to get CI

$$\begin{aligned} C(\mathbf{X}) &= \{\mu : \mathbf{X} \in A(\mu)\} \\ &= \left\{ \mu : \frac{\overline{X} - \mu}{s_{\mathbf{X}}/\sqrt{n}} \geq -t_{n-1,\alpha} \right\} \\ &= \left\{ \mu : \overline{X} - \mu \geq -\frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha} \right\} \\ &= \left\{ \mu : \mu \leq \overline{X} + \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha} \right\} \\ &= \left(-\infty, \overline{X} + \frac{s_{\mathbf{X}}}{\sqrt{n}} t_{n-1,\alpha} \right] \end{aligned}$$

Example - lower bounded CI - solution

LRT level α test reject H_0 if and only if

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Example

Problem

$X_1, \cdots, X_n$ are iid samples from a distribution with mean μ and finite variance σ^2. Construct asymptotic $(1-\alpha)$ two-sided interval for μ

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Solution

Let $\overline{X}$ be a method of moment estimator for μ.
By law of large number, $\overline{X}$ is consistent for μ, and by central limit theorem,

$$\overline{X} \sim \mathcal{AN}\left(\mu, \frac{\sigma^2}{n}\right)$$

Example (cont'd)

Consider testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$. The Wald statistic

$$Z_n = \frac{\overline{X} - \mu_0}{S_n}$$

for a consistent estimator of $\sigma/\sqrt{n}$.

Example (cont'd)

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The Wald level α test

$$\left| \frac{(\overline{X} - \mu_0)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}} \right| > z_{\alpha/2}$$

Example (cont'd)

The acceptance region is

$$A(\mu_0) = \left\{ \mathbf{x} : \left| \frac{(\overline{x} - \mu_0)\sqrt{n}}{\sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}} \right| \leq z_{\alpha/2} \right\}$$

Example (cont'd)

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$(1-\alpha)$ CI is

$$C(\mathbf{x}) = \{\mu : \mathbf{x} \in A(\mu)\}$$

Example (cont'd)

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Example (cont'd)

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Summary

Today

- Interval Estimation
- Confidence Interval

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Next Lectures

- Reviews and Example Problems (every lecture)
- E-M algorithm
- Non-informative priors
- Bayesian Tests