Biostatistics 602 - Statistical Inference
Lecture 10
Maximum Likelihood Estimator

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February 12th, 2013

Last Lecture

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2. What is a method of moment estimator?
3. What are advantages and disadvantages of method of moment estimator?
4. What is a maximum likelihood estimator (MLE)?
5. How can you find an MLE?

Recap - Method of Moment Estimator

- Point Estimation - Estimate θ or $\tau(\theta)$.
- Method of Moment

$$m_1 = \frac{1}{n}\sum X_i = E\mathbf{X} = \mu_1$$
$$m_2 = \frac{1}{n}\sum X_i^2 = E\mathbf{X}^2 = \mu_2$$
$$\vdots$$
$$m_k = \frac{1}{n}\sum X_i^k = E\mathbf{X}^k = \mu_k$$

Recap - Example of Method of Moment Estimator

$$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$$

$$\hat{\mu} = \overline{X}$$

$$\hat{\mu}^2 + \hat{\sigma}^2 = E\mathbf{X}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\hat{\sigma}^2 = \sum (X_i - \overline{X})^2 / n$$

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- Easy to understand
- Estimators can be improved; use as initial value to get other estimators
- No guarantee that the estimator will fall into the range of valid parameter space.

Recap - Likelihood Function

Definition

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} f_X(x|\theta)$. The join distribution of $\mathbf{X} = (X_1, \cdots, X_n)$ is

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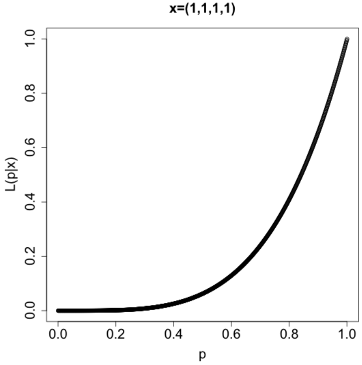
Given that $\mathbf{X} = \mathbf{x}$ is observed, the function of θ defined by $L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$ is called the likelihood function.

Recap - Example Likelihood Function

- $X_1, X_2, X_3, X_4 \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p),\ 0 < p < 1.$
- $\mathbf{x} = (1, 1, 1, 1)^T$
- Intuitively, it is more likely that p is larger than smaller.
- $L(p|\mathbf{x}) = f(\mathbf{x}|p) = \prod_{i=1}^{4} p^{x_i}(1-p)^{1-x_i} = p^4.$



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     (a) $\partial^2 L(\theta_1, \theta_2)^2 / \partial\theta_1^2 < 0$ or $\partial^2 L(\theta_1, \theta_2)^2 / \partial\theta_2^2 < 0$.

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Example of MLE : Uniform Distribution

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We need to maximize $1/\theta^n$ subject to constraint that $0 \leq x_{(n)} \leq \theta$. Because $1/\theta^n$ decreases in θ, the MLE is $\hat{\theta}(\mathbf{X}) = X_{(n)}$.

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Suppose n pairs of data $(X_1, Y_1), \cdots, (X_n, Y_n)$ where X_i is generated from an unknown distribution, and Y_i are generated conditionally on X_i.

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Find the MLE of $(\alpha, \beta, \sigma^2)$.

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Solution : Normal Distribution (cont'd)

The likelihood function is

$$L(\alpha, \beta, \sigma^2 | \mathbf{x}, \mathbf{y}) = f_{\mathbf{X}}(\mathbf{x})(2\pi\sigma^2)^{-n/2} \exp\left[-\frac{\sum_{i=1}^n (y_i - \alpha - \beta x_i)^2}{2\sigma^2}\right]$$

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$$l(\alpha, \beta, \sigma^2) = C - \frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^n (y_i - \alpha - \beta x_i)^2}{2\sigma^2}$$

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$$\frac{\partial l}{\partial \beta} = \frac{2\sum_{i=1}^{n}(y_i - \alpha - \beta x_i)x_i}{2\sigma^2} = \frac{\sum_{i=1}^{n} x_i y_i - n\alpha\bar{x} - \beta\sum_{i=1}^{n} x_i^2}{\sigma^2} = 0$$

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$$L(\mu|\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2}\right] = (2\pi)^{-n/2} \exp\left[-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2}\right]$$

$$l(\mu|\mathbf{x}) = \log L(\mu, \mathbf{x}) = C - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2}$$

$$\frac{\partial l}{\partial \mu} = \frac{2\sum_{i=1}^{n}(x_i-\mu)}{2} = 0, \qquad \frac{\partial^2 l}{\partial \mu^2} < 0$$

Example : Normal Distribution with Known Variance

Problem

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Are we done?

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$$\hat{\mu}(\mathbf{X}) = \max(\overline{X}, 0)$$

Invariance Property of MLE

Question

If $\hat{\theta}$ is the MLE of θ, what is the MLE of $\tau(\theta)$?

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1. What is the MLE of p?

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Example

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1. What is the MLE of p?
2. What is the MLE of odds, defined by $\eta = p/(1-p)$?

MLE of p

$$L(p|\mathbf{x}) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

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- Equivalently, $L^*(\eta|\mathbf{x})$ is maximized when $\eta = \hat{p}/(1-\hat{p}) = \tau(\hat{p})$, because τ is a one-to-one function.
- Therefore $\hat{\eta} = \tau(\hat{p})$.

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Fact

Denote the MLE of θ by $\hat{\theta}$. If $\tau(\theta)$ is an one-to-one function of θ, then MLE of $\tau(\theta)$ is $\tau(\hat{\theta})$.

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The likelihood function in terms of $\tau(\theta) = \eta$ is

$$L^*(\tau(\theta)|\mathbf{x}) = \prod_{i=1}^{n} f_X(x_i|\theta) = \prod_{i=1}^{n} f(x_i|\tau^{-1}(\eta))$$

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We know this function is maximized when $\tau^{-1}(\eta) = \hat{\theta}$, or equivalently, when $\eta = \tau(\hat{\theta})$. Therefore, MLE of $\eta = \tau(\theta)$ is $\tau(\hat{\theta})$.

Summary

Today

- Maximum Likelihood Estimator

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Next Lecture

- Mean Squared Error
- Unbiased Estimator
- Cramer-Rao inequality