Biostatistics 602 - Statistical Inference
Lecture 14
Obtaining Best Unbiased Estimator

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Last Lecture

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- What is the Rao-Blackwell Theorem?
- Is the best unbiased estimator (UMVUE) for $\tau(\theta)$ unique?
- What is the relationship between the UMVUE and the unbiased estimators of zero?

Rao-Blackwell Theorem

Theorem 7.3.17

Let $W(\mathbf{X})$ be any unbiased estimator of $\tau(\theta)$, and T be a sufficient statistic for θ. Define $\phi(T) = E[W|T]$. Then the followings hold.

1. $E[\phi(T)|\theta] = \tau(\theta)$
2. $\text{Var}[\phi(T)|\theta] \leq \text{Var}(W|\theta)$ for all θ.

That is, $\phi(T)$ is a uniformly better unbiased estimator of $\tau(\theta)$.

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Theorem 7.3.20 - UMVUE and unbiased estimators of zero

If $E[W(\mathbf{X})] = \tau(\theta)$. W is the best unbiased estimator of $\tau(\theta)$ if an only if W is uncorrelated with all unbiased estimator of 0.

The power of complete sufficient statistics

Theorem 7.3.23

Let T be a complete sufficient statistic for parameter θ. Let $\phi(T)$ be any estimator based on T. Then $\phi(T)$ is the unique best unbiased estimator of its expected value.

Remarks from previous Theorems - #1

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- $W(\mathbf{X})$: unbiased for $\tau(\theta)$.
- $T^*(\mathbf{X})$: sufficient statistic for θ.

$\phi(T) = E[W(\mathbf{X})|T(\mathbf{X})]$ is a better unbiased estimator of $\tau(\theta)$.

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In fact, we only need to consider functions of minimal sufficient statistics to find the best unbiased estimator.

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In fact, we only need to consider functions of minimal sufficient statistics to find the best unbiased estimator.

Let $T(\mathbf{X})$ be a minimal sufficient, and $T^*(\mathbf{X})$ be a sufficient statistic. Then by definition, there exists a function h that satisfies $T = h(T^*)$.

$$\begin{aligned} E[\phi(T)|T^*] &= E[\phi\{h(T^*)\}|T^*] \\ &= \phi\{h(T^*)\} = \phi(T) \end{aligned}$$

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$$\begin{aligned} E[\phi(T)|T^*] &= E[\phi\{h(T^*)\}|T^*] \\ &= \phi\{h(T^*)\} = \phi(T) \end{aligned}$$

Therefore $\phi(T)$ remains the same after conditioning on any sufficient statistic T^*.

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- By Theorem 7.3.20, $\phi(T)$ is the best unbiased estimator if and only if $\phi(T)$ if and only of $\phi(T)$ is uncorrelated with $U(T)$, which is any unbiased esimator of 0.

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- By definition, T is complete is $E[U(T)] = 0$ for all θ implies $U(T) = 0$ almost surely.
- Suppose that T is a complete statistic, then $U(T)$ can only be zero almost surely.
- Therefore, $\text{Cov}(\phi(T), U(T)) = \text{Cov}(\phi(T), 0) = 0$, and $\phi(T)$ is the best unbiased estimator of its expected value (Theorem 7.3.23).

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   - Guess a functon $\phi(T)$ such that $E[\phi(T)] = \tau(\theta)$.
   - Guess an unbiased estimator $h(\mathbf{X})$ of $\tau(\theta)$. Construct $\phi(T) = E[h(\mathbf{X})|T]$, then $E[\phi(T)] = E[h(\mathbf{X})] = \tau(\theta)$.

Example - Normal Distribution

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$. Find the best unbiased estimator for (1) μ, (2) σ^2, (3) μ^2.

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- We know that $\mathbf{T}(\mathbf{X}) = (\overline{X}, s^2_{\mathbf{X}})$ is complete, sufficient statistic for (μ, σ^2).
- Because $E[\overline{X}] = \mu$, $\overline{X}$ is an unbiased estimator for μ, $\overline{X}$ is also a function of $\mathbf{T}(\mathbf{X})$.
- Therefore, $\overline{X}$ is the best unbiased estimator for μ.

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- Therefore $s^2_{\mathbf{X}}$ is the best unbiased estimator of σ^2.

Example - Normal Distribution (cont'd)

To obtain UMVUE for μ^2, we need a $\phi(\mathbf{T}) = \phi(\overline{X}, s^2_{\mathbf{X}})$ such that $E[\phi(\mathbf{T})] = \mu^2$.

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$$E(\overline{X}) = \mu$$

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- $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is unbiased estimator for μ^2
- And it is a function of $(\overline{X}, s^2_{\mathbf{X}})$.
- Hence, $\overline{X}^2 - s^2_{\mathbf{X}}/n$ is the best unbiased estimator for μ^2.

Example - Normal Distribution - Alternative method

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&= \frac{E[(\sum_{i=1}^n X_i)^2\,|\mathbf{T}] - E[\sum_{i=1}^n X_i^2|\mathbf{T}]}{n(n-1)} \\
&= \frac{E[\left(n\overline{X}\right)^2 - (n-1)s_{\mathbf{X}}^2 - n\overline{X}^2|\mathbf{T}]}{n(n-1)}
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&= \overline{X}^2 - s_{\mathbf{X}}^2 / n
\end{aligned}$$

Example - Uniform Distribution

Problem

$X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$. Find the best unbiased estimator for (1) θ, (2) $g(\theta)$ differentiable on $(0, \theta)$ (3) θ^2, (4) $1/\theta$.

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Solution - MVUE of θ

- $T(\mathbf{X}) = X_{(n)}$ is a complete and sufficient statistic for θ.

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- $E[\phi(T)] = E\left[\frac{n+1}{n} X_{(n)}\right] = \theta$.

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- $E[\phi(T)] = E\left[\frac{n+1}{n} X_{(n)}\right] = \theta$.

$\frac{n+1}{n} X_{(n)}$ is the best unbiased estimator of θ.

Example - Uniform Distribution - for $g(\theta)$

We need to find a function of $\phi(T) = X_{(n)}$ such that $E[\phi(T)] = g(\theta)$.

$$g(\theta) \quad = \quad E[\phi(T)] = \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt$$

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Taking derivative with respect to θ, and applying Leibnitz's rule.

$$g'(\theta) = \frac{d}{d\theta} \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt$$

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$$\begin{aligned} g'(\theta) &= \frac{d}{d\theta} \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt \\ &= \phi(\theta) n\theta^{-n}\theta^{n-1} + \int_0^\theta \phi(t) t^{n-1} n \frac{d}{d\theta}\theta^{-n} dt \end{aligned}$$

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$$g(\theta) = E[\phi(T)] = \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt$$

Taking derivative with respect to θ, and applying Leibnitz's rule.

$$\begin{aligned} g'(\theta) &= \frac{d}{d\theta} \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt \\ &= \phi(\theta) n\theta^{-n}\theta^{n-1} + \int_0^\theta \phi(t) t^{n-1} n \frac{d}{d\theta}\theta^{-n} dt \\ &= \phi(\theta) n\theta^{-1} + \int_0^\theta \phi(t) t^{n-1} n(-n)\theta^{-n-1} dt \end{aligned}$$

Example - Uniform Distribution - for $g(\theta)$

We need to find a function of $\phi(T) = X_{(n)}$ such that $E[\phi(T)] = g(\theta)$.

$$g(\theta) = E[\phi(T)] = \int_0^\theta \phi(t) n\theta^{-n} t^{n-1} dt$$

Taking derivative with respect to θ, and applying Leibnitz's rule.

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Example - Uniform Distribution - for $g(\theta)$ (cont'd)

$$g'(\theta) = \phi(\theta) n\theta^{-1} - n\theta^{-1} \int_0^\theta \phi(t) n t^{n-1} \theta^{-n} dt$$

Example - Uniform Distribution - for $g(\theta)$ (cont'd)

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&= \phi(\theta) n\theta^{-1} - n\theta^{-1} g(\theta) \\
\phi(\theta) &= \frac{g'(\theta) + n\theta^{-1} g(\theta)}{n\theta^{-1}}
\end{aligned}
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Therefore, the best unbiased estimator of $g(\theta)$ is

$$
\phi(T) = \frac{g'(T) + nT^{-1} g(T)}{nT^{-1}}
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$$\begin{aligned} \phi(T) &= \frac{g'(T) + nT^{-1}g(T)}{nT^{-1}} \\ \phi(X_{(n)}) &= \frac{g'(X_{(n)}) + nX_{(n)}^{-1}g(X_{(n)})}{nX_{(n)}^{-1}} \end{aligned}$$

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Therefore, the best unbiased estimator of $g(\theta)$ is

$$\begin{aligned} \phi(T) &= \frac{g'(T) + nT^{-1}g(T)}{nT^{-1}} \\ \phi(X_{(n)}) &= \frac{g'(X_{(n)}) + nX_{(n)}^{-1}g(X_{(n)})}{nX_{(n)}^{-1}} \\ &= \frac{1}{n}X_{(n)}g'(X_{(n)}) + g(X_{(n)}) \end{aligned}$$

Example - Uniform Distribution - for θ^2

$g(\theta) = \theta^2$, and $g'(\theta) = 2\theta$.

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$$\begin{aligned}
\phi(X_{(n)}) &= \frac{1}{n} X_{(n)} \cdot 2X_{(n)} + X_{(n)}^2 \\
&= \frac{n+2}{n} X_{(n)}^2
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Example - Uniform Distribution - for $1/\theta$

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$$\begin{aligned} \phi(X_{(n)}) &= \frac{1}{n} X_{(n)} \cdot \left(-\frac{1}{X_{(n)}^2} \right) + \frac{1}{X_{(n)}} \\ &= \frac{n-1}{nX_{(n)}} \end{aligned}$$

Example - Binomial best unbiased estimator

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Binomial}(k, \theta)$. Estimate the probability of exactly one success.

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- We know that $T(\mathbf{X}) = \sum_{i=1}^n X_i \sim \text{Binomial}(kn, \theta)$ and it is a complete sufficient statistic.
- So we need to find a $\phi(T)$ that satisfies $E[\phi(T)] = \tau(\theta)$.
- There is no imeediately evident unbiased estimator of $\tau(\theta)$ as a function of T.

Solution - Binomial best unbiased estimator

- Start with a simple-minded estimator

$$W(\mathbf{X}) = \begin{cases} 1 & X_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

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and hence is an unbiased estimator of $\tau(\theta) = k\theta(1-\theta)^{k-1}$.
- The best unbiased estimator of $\tau(\theta)$ is

$$\phi(T) = E[W|T] = E[W(\mathbf{X})|T(\mathbf{X})]$$

Solution - Binomial best unbiased estimator (cont'd)

$$\phi(t) = E\left[W(\mathbf{X})|\sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1|\sum_{i=1}^{n} X_i = t\right]$$

Solution - Binomial best unbiased estimator (cont'd)

$$\phi(t) = E\left[W(\mathbf{X})|\sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1|\sum_{i=1}^{n} X_i = t\right]$$

$$= \frac{\Pr(X_1 = 1, \sum_{i=1}^{n} X_i = t)}{\Pr(\sum_{i=1}^{n} X_i = t)}$$

Solution - Binomial best unbiased estimator (cont'd)

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\begin{aligned}
\phi(t) &= E\left[W(\mathbf{X})|\sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1|\sum_{i=1}^{n} X_i = t\right] \\
&= \frac{\Pr(X_1 = 1, \sum_{i=1}^{n} X_i = t)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\Pr(X_1 = 1, \sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)}
\end{aligned}
$$

Solution - Binomial best unbiased estimator (cont'd)

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\begin{aligned}
\phi(t) &= E\left[W(\mathbf{X}) | \sum_{i=1}^n X_i = t\right] = \Pr\left[X_1 = 1 | \sum_{i=1}^n X_i = t\right] \\
&= \frac{\Pr(X_1 = 1, \sum_{i=1}^n X_i = t)}{\Pr(\sum_{i=1}^n X_i = t)} \\
&= \frac{\Pr(X_1 = 1, \sum_{i=2}^n X_i = t-1)}{\Pr(\sum_{i=1}^n X_i = t)} \\
&= \frac{\Pr(X_1 = 1)\Pr(\sum_{i=2}^n X_i = t-1)}{\Pr(\sum_{i=1}^n X_i = t)}
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Solution - Binomial best unbiased estimator (cont'd)

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&= \frac{\Pr(X_1 = 1, \sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\Pr(X_1 = 1)\Pr(\sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{[k\theta(1-\theta)^{k-1}]\left[\binom{k(n-1)}{t-1}\theta^{t-1}(1-\theta)^{k(n-1)-t-1}\right]}{\binom{kn}{n}\theta^{t}(1-\theta)^{kn-t}}
\end{aligned}$$

Solution - Binomial best unbiased estimator (cont'd)

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\begin{aligned}
\phi(t) &= E\left[W(\mathbf{X}) | \sum_{i=1}^{n} X_i = t\right] = \Pr\left[X_1 = 1 | \sum_{i=1}^{n} X_i = t\right] \\
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&= \frac{\Pr(X_1 = 1, \sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{\Pr(X_1 = 1) \Pr(\sum_{i=2}^{n} X_i = t-1)}{\Pr(\sum_{i=1}^{n} X_i = t)} \\
&= \frac{[k\theta(1-\theta)^{k-1}]\left[\binom{k(n-1)}{t-1}\theta^{t-1}(1-\theta)^{k(n-1)-t-1}\right]}{\binom{kn}{n}\theta^t(1-\theta)^{kn-t}} = k\frac{\binom{k(n-1)}{t-1}}{\binom{kn}{t}}
\end{aligned}
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Solution - Binomial best unbiased estimator (cont'd

Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

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Therefore, the unbiased estimator of $k\theta(1-\theta)^{k-1}$ is

$$\phi\left(\sum_{i=1}^{n} X_i\right) = k\frac{\binom{k(n-1)}{\sum X_i - 1}}{\binom{kn}{\sum X_i}}$$

Summary

Today

- Rao-Blackwell Theorem
- Methods for obtaining UMVUE

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- Rao-Blackwell Theorem
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Next Lecture

- Bayesian Estimators