Biostatistics 602 - Statistical Inference
Lecture 24
E-M Algorithm & Practice Examples

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April 16th, 2013

Last Lecture

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- What is an interval estimator?

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- What is the coverage probability, confidence coefficient, and confidence interval?
- How can a $1 - \alpha$ confidence interval typically be constructed?
- To obtain a lower-bounded (upper-tail) CI, whose acceptance region of a test should be inverted?
  (a) $H_0 : \theta = \theta_0$ vs $H_1 : \theta > \theta_0$
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where $\mathbf{X}$ are random samples from $f_{\mathbf{X}}(\mathbf{x}|\theta)$. In other words, it is the average length of the interval estimator.

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2. To obtain a lower-bounded CI $[L(\mathbf{X}), \infty)$, then we invert the acceptance region of a test for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$, where $\Omega = \{\theta : \theta \geq \theta_0\}$.

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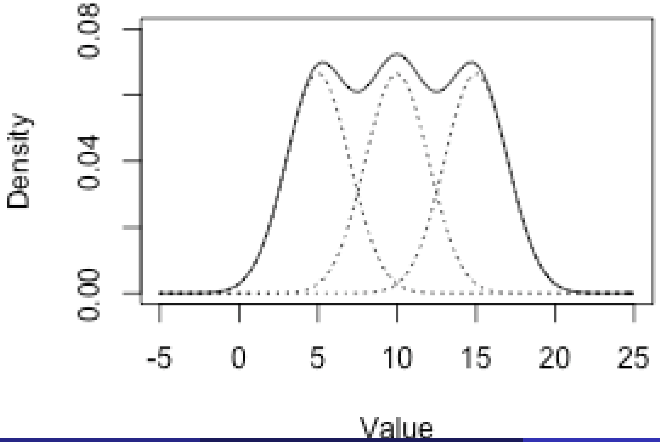
4. Check boundary points to see whether boundary gives global maximum.

Example: A mixture distribution



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- k number of mixture components

MLE Problem for mixture of normals

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Find MLEs for $\theta = (\pi, \mu, \sigma^2)$.

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- $\pi = \pi_1 = 1$
- $\mu = \mu_1 = \overline{x}$
- $\sigma^2 = \sigma_1^2 = \sum_{i=1}^{n} (x_i - \overline{x})^2 / n$

Incomplete data problem when $k > 1$

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The MLE solution is not analytically tractable, because it involves multiple sums of exponential functions.

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The MLE solution is analytically tractable, if **z** is known.

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The algorithm was derived and used in various special cases by a number of authors, but it was not identified as a general algorithm until the seminal paper by Dempster, Laird, and Rubin in Journal of Royal Statistical Society Series B (1977).

Overview of E-M Algorithm

Basic Structure

- $\mathbf{y}$ is observed (or incomplete) data
- $\mathbf{z}$ is missing (or augmented) data
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We are interested in MLE for $L(\theta|\mathbf{y}) = g(\mathbf{y}|\theta)$.

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Iteratively maximizing the first term in the right-hand side results in E-M algorithm.

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$$Q(\theta|\theta^{(r)}) \quad = \quad \mathrm{E}\left[\log f(\mathbf{y},\mathbf{Z}|\theta)|\mathbf{y},\theta^{(r)}\right]$$

where $\theta^{(r)}$ is the estimation of θ in r-th iteration.

Overview of E-M Algorithm (cont'd)

Objective

- Maximize $L(\theta|\mathbf{y})$ or $l(\theta|\mathbf{y})$.
- Let $f(\mathbf{y}, \mathbf{z}|\theta)$ denotes the pdf of complete data. In E-M algorithm, rather than working with $l(\theta|\mathbf{y})$ directly, we work with the surrogate function

$$Q(\theta|\theta^{(r)}) = \mathrm{E}\left[\log f(\mathbf{y}, \mathbf{Z}|\theta)|\mathbf{y}, \theta^{(r)}\right]$$

  where $\theta^{(r)}$ is the estimation of θ in r-th iteration.
- $Q(\theta|\theta^{(r)})$ is the *expected log-likelihood of complete data*, conditioning on the observed data and $\theta^{(r)}$.

Key Steps of E-M algorithm

Expectation Step

- Compute $Q(\theta|\theta^{(r)})$.
- This typically involves in estimating the conditional distribution $\mathbf{Z}|\mathbf{Y}$, assuming $\theta = \theta^{(r)}$.
- After computing $Q(\theta|\theta^{(r)})$, move to the M-step

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Maximization Step

- Maximize $Q(\theta|\theta^{(r)})$ with respect to θ.
- The $\arg\max_\theta Q(\theta|\theta^{(r)})$ will be the $(r+1)$-th θ to be fed into the E-step.
- Repeat E-step until convergence

E-M algorithm for mixture of normals

E-step

$$Q(\theta|\theta^{(r)}) = \mathrm{E}\left[\log f(\mathbf{y}, \mathbf{Z}|\theta)|\mathbf{y}, \theta^{(r)}\right]$$

E-M algorithm for mixture of normals

E-step

$$\begin{aligned} Q(\theta|\theta^{(r)}) &= \mathrm{E}\left[\log f(\mathbf{y},\mathbf{Z}|\theta)|\mathbf{y},\theta^{(r)}\right] \\ &= \sum_{\mathbf{z}} k(\mathbf{z}|\theta^{(r)},\mathbf{y}) \log f(\mathbf{y},\mathbf{z}|\theta) \end{aligned}$$

E-M algorithm for mixture of normals

E-step

$$
\begin{aligned}
Q(\theta|\theta^{(r)}) &= \mathrm{E}\left[\log f(\mathbf{y}, \mathbf{Z}|\theta)|\mathbf{y}, \theta^{(r)}\right] \\
&= \sum_{\mathbf{z}} k(\mathbf{z}|\theta^{(r)}, \mathbf{y}) \log f(\mathbf{y}, \mathbf{z}|\theta) \\
&= \sum_{i=1}^{n} \sum_{z_i=1}^{k} k(z_i|\theta^{(r)}, y_i) \log f(y_i, z_i|\theta)
\end{aligned}
$$

E-M algorithm for mixture of normals

E-step

$$
\begin{aligned}
Q(\theta|\theta^{(r)}) &= \mathrm{E}\left[\log f(\mathbf{y},\mathbf{Z}|\theta)|\mathbf{y},\theta^{(r)}\right] \\
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&= \sum_{i=1}^{n}\sum_{z_i=1}^{k} \frac{f(y_i,z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})} \log f(y_i,z_i|\theta)
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E-M algorithm for mixture of normals

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Q(\theta|\theta^{(r)}) &= \mathrm{E}\left[\log f(\mathbf{y}, \mathbf{Z}|\theta)|\mathbf{y}, \theta^{(r)}\right] \\
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&= \sum_{i=1}^{n} \sum_{z_i=1}^{k} \frac{f(y_i, z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})} \log f(y_i, z_i|\theta) \\
f(y_i, z_i|\theta) &\sim \mathcal{N}(\mu_{z_i}, \sigma^2_{z_i})
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$$

E-M algorithm for mixture of normals

E-step

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\begin{aligned}
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&= \sum_{i=1}^{n} \sum_{z_i=1}^{k} \frac{f(y_i, z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})} \log f(y_i, z_i|\theta) \\
f(y_i, z_i|\theta) &\sim \mathcal{N}(\mu_{z_i}, \sigma^2_{z_i}) \\
g(y_i|\theta) &= \sum_{j=1}^{k} \pi_i f(y_i, z_i = j|\theta)
\end{aligned}
$$

E-M algorithm for mixture of normals (cont'd)

M-step

$$Q(\theta|\theta^{(r)}) = \sum_{i=1}^{n} \sum_{z_i=1}^{k} \frac{f(y_i, z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})} \log f(y_i, z_i|\theta)$$

E-M algorithm for mixture of normals (cont'd)

M-step

$$Q(\theta|\theta^{(r)}) = \sum_{i=1}^{n}\sum_{z_i=1}^{k}\frac{f(y_i, z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})}\log f(y_i, z_i|\theta)$$

$$\pi_j^{(r+1)} = \frac{1}{n}\sum_{i=1}^{n} k(z_i = j|y_i, \theta^{(r)}) = \frac{1}{n}\sum_{i=1}^{n}\frac{f(y_i, z_i = j|\theta^{(r)})}{g(y_i|\theta^{(r)})}$$

E-M algorithm for mixture of normals (cont'd)

M-step

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\mu_j^{(r+1)} &= \frac{\sum_{i=1}^{n} x_i k(z_i = j|y_i, \theta^{(r)})}{\sum_{i=1}^{n} k(z_i = j|y_i, \theta^{(r)})} = \frac{\sum_{i=1}^{n} x_i k(z_i = j|y_i, \theta^{(r)})}{n\pi_j^{(r+1)}}
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$$

E-M algorithm for mixture of normals (cont'd)

M-step

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\begin{aligned}
Q(\theta|\theta^{(r)}) &= \sum_{i=1}^{n}\sum_{z_i=1}^{k}\frac{f(y_i,z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})}\log f(y_i,z_i|\theta) \\
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\sigma_j^{2,(r+1)} &= \frac{\sum_{i=1}^{n}(x_i-\mu_j^{(r+1)})^2 k(z_i=j|y_i,\theta^{(r)})}{\sum_{i=1}^{n}k(z_i=j|y_i,\theta^{(r)})}
\end{aligned}
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E-M algorithm for mixture of normals (cont'd)

M-step

$$Q(\theta|\theta^{(r)}) = \sum_{i=1}^{n}\sum_{z_i=1}^{k}\frac{f(y_i, z_i|\theta^{(r)})}{g(y_i|\theta^{(r)})}\log f(y_i, z_i|\theta)$$

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$$\mu_j^{(r+1)} = \frac{\sum_{i=1}^{n}x_i k(z_i=j|y_i,\theta^{(r)})}{\sum_{i=1}^{n}k(z_i=j|y_i,\theta^{(r)})} = \frac{\sum_{i=1}^{n}x_i k(z_i=j|y_i,\theta^{(r)})}{n\pi_j^{(r+1)}}$$

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Does E-M iteration converge to MLE?

Theorem 7.2.20 - Monotonic EM sequence

The sequence $\{\hat{\theta}^{(r)}\}$ defined by the E-M procedure satisfies

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$$E\left[\log L\left(\hat{\theta}^{(r+1)}|\mathbf{y},\mathbf{Z}\right)|\hat{\theta}^{(r)},\mathbf{y}\right] \quad = \quad E\left[\log L\left(\hat{\theta}^{(r)}|\mathbf{y},\mathbf{Z}\right)|\hat{\theta}^{(r)},\mathbf{y}\right]$$

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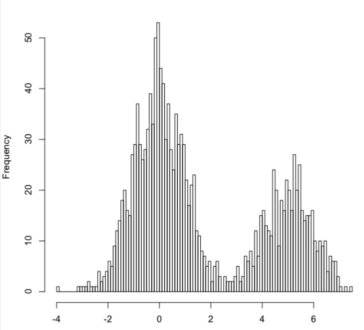
Theorem 7.5.2 further guarantees that $L(\hat{\theta}^{(r)}|\mathbf{y})$ converges monotonically to $L(\hat{\theta}|\mathbf{y})$ for some stationary point $\hat{\theta}$.

A working example (from BIOSTAT615/815 Fall 2012)

Example Data (n=1,500)



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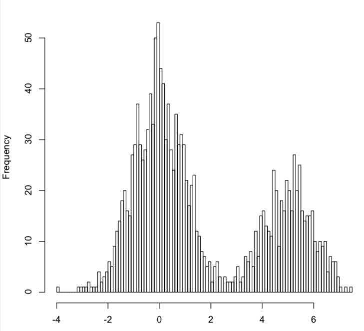

Running example of implemented software

```
user@host~/> ./mixEM ./mix.dat
Maximum log-likelihood = 3043.46, at pi = (0.667842,0.332158)
between N(-0.0299457,1.00791) and N(5.0128,0.913825)
```

Practice Problem 1

Problem

Let $X_1, \cdots, X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta} \qquad -\theta < x < \theta,\ \theta > 0$$

Find, if one exists, a best unbiased estimator of θ.

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Strategy to solve the problem

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  1. Find a complete sufficient statistic T.
  2. For a trivial unbiased estimator W for θ, and compute $\phi(T) = \mathrm{E}[W|T]$
  3. or Make a function $\phi(T)$ such that $\mathrm{E}[\phi(T)] = \theta$.

Solution

First, we need to find a complete sufficient statistic.

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$$
\begin{aligned}
f_X(x|\theta) &= \frac{1}{2\theta} I(|x| < \theta) \\
f_{\mathbf{X}}(\mathbf{x}|\theta) &= \frac{1}{(2\theta)^n} I(\max_i |x_i| < \theta)
\end{aligned}
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Let $T(\mathbf{X}) = \max_i |X_i|$, then $f_T(t|\theta) = \frac{nt^{n-1}}{\theta^n} I(0 \leq t < \theta)$

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Let $T(\mathbf{X}) = \max_i |X_i|$, then $f_T(t|\theta) = \frac{nt^{n-1}}{\theta^n} I(0 \leq t < \theta)$

$$E[g(T)] = \int_0^\theta \frac{nt^{n-1} g(t)}{\theta^n} dt = 0$$

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$$\int_0^\theta t^{n-1} g(t)\, dt = 0$$
$$\theta^{n-1} g(\theta) = 0$$

Solution

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Let $T(\mathbf{X}) = \max_i |X_i|$, then $f_T(t|\theta) = \frac{nt^{n-1}}{\theta^n} I(0 \leq t < \theta)$

$$\begin{aligned} E[g(T)] &= \int_0^\theta \frac{nt^{n-1} g(t)}{\theta^n} dt = 0 \\ \int_0^\theta t^{n-1} g(t)\, dt &= 0 \\ \theta^{n-1} g(\theta) &= 0 \\ g(\theta) &= 0 \end{aligned}$$

Therefore the family of T is complete.

Solution

We need to make a $\phi(T)$ such that $E[\phi(T)] = \theta$.

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$$\begin{aligned} E[T] &= \int_0^\theta t \frac{nt^{n-1}}{\theta^n} dt \\ &= \int_0^\theta \frac{nt^n}{\theta^n} dt \end{aligned}$$

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$$\begin{aligned} E[T] &= \int_0^\theta t \frac{nt^{n-1}}{\theta^n} dt \\ &= \int_0^\theta \frac{nt^n}{\theta^n} dt \\ &= \frac{n}{n+1}\theta \end{aligned}$$

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$\phi(T) = \frac{n+1}{n} T$ is an unbiased estimator and a function of a complete sufficient statistic.

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$\phi(T) = \frac{n+1}{n} T$ is an unbiased estimator and a function of a complete sufficient statistic.
Therefore, $\phi(T)$ is the best unbiased estimator by Theorem 7.3.23.

Practice Problem 2

Problem

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the probability that the first n observations exceed the $(n+1)$-st.

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1. Show that

$$W(X_1, \cdots, X_{n+1}) = I\left(\sum_{i=1}^{n} X_i > X_{n+1}\right)$$

is an unbiased estimator of $h(p)$.

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1. Show that

$$W(X_1, \cdots, X_{n+1}) = I\left(\sum_{i=1}^{n} X_i > X_{n+1}\right)$$

   is an unbiased estimator of $h(p)$.

2. Find the best unbiased estimator of $h(p)$.

Solution for (a)

$$E[W] = \sum_{\mathbf{X}} W(\mathbf{X}) \Pr(\mathbf{X})$$

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E[W] &= \sum_{\mathbf{X}} W(\mathbf{X}) \Pr(\mathbf{X}) \\
&= \sum_{\mathbf{X}} I\left(\sum_{i=1}^{n} X_i > X_{n+1}\right) \Pr(\mathbf{X})
\end{aligned}
$$

Solution for (a)

$$
\begin{aligned}
E[W] &= \sum_{\mathbf{X}} W(\mathbf{X}) \Pr(\mathbf{X}) \\
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&= \sum_{\sum_{i=1}^{n} X_i > X_{n+1}} \Pr(\mathbf{X})
\end{aligned}
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E[W] &= \sum_{\mathbf{X}} W(\mathbf{X}) \Pr(\mathbf{X}) \\
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Solution for (a)

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\end{aligned}$$

Therefore T is an unbiased estimator of $h(p)$.

Solution for (b)

$T = \sum_{i=1}^{n+1} X_i$ is complete sufficient statistic for p.

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$T = \sum_{i=1}^{n+1} X_i$ is complete sufficient statistic for p.

$$\begin{aligned} \phi(T) &= E[W|T] = \Pr(W = 1|T) \\ &= \Pr\left(\sum_{i=1}^{n} X_i > X_{n+1} | T\right) \end{aligned}$$

Solution for (b)

$T = \sum_{i=1}^{n+1} X_i$ is complete sufficient statistic for p.

$$\begin{aligned} \phi(T) &= E[W|T] = \Pr(W = 1|T) \\ &= \Pr\left(\sum_{i=1}^{n} X_i > X_{n+1}|T\right) \end{aligned}$$

- If $T = 0$, then $\sum_{i=1}^{n} X_i = X_{n+1}$

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- If $T = 2$ then
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  - $\Pr(\sum_{i=1}^{n} X_i = 1 = X_{n+1} = 1) = 2/(n+1)$
- If $T > 2$, then $\sum_{i=1}^{n} X_i \geq 2 > 1 \geq X_{n+1}$

Solution for (b) (cont'd)

Therefore, the best unbiased estimator is

$$\begin{aligned}\phi(T) &= \Pr\left(\sum_{i=1}^{n} X_i > X_{n+1} | T\right) \\ &= \begin{cases} 0 & T = 0 \\ n/(n+1) & T = 1 \\ (n-1)/(n+1) & T = 2 \\ 1 & T \geq 3 \end{cases}\end{aligned}$$

Practice Problem 3

Problem

Suppose $X_1, \cdots, X_n$ are iid samples from $f(x|\theta) = \theta \exp(-\theta x)$. Suppose the prior distribution of θ is

$$\pi(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta}$$

where α, β are known.

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where α, β are known.

(a) Derive the posterior distribution of θ.

(b) If we use the loss function $L(\theta, a) = (a - \theta)^2$, what is the Bayes rule estimator for θ?

(a) Posterior distribution of θ

$$
\begin{aligned}
f(\mathbf{x}, \theta) &= \pi(\theta) f(\mathbf{x}|\theta)\pi(\theta) \\
&= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \theta^{\alpha-1} e^{-\theta/\beta} \prod_{i=1}^{n} \left[\theta \exp\left(-\theta x_i\right)\right]
\end{aligned}
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&= \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha+n-1}\exp\left[-\theta\left(1/\beta+\sum_{i=1}^{n}x_i\right)\right]
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&\propto \text{Gamma}\left(\alpha+n-1, \frac{1}{\beta^{-1}+\sum_{i=1}^{n}x_i}\right)
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\pi(\theta|\mathbf{x}) &= \text{Gamma}\left(\alpha+n-1, \frac{1}{\beta^{-1}+\sum_{i=1}^{n}x_i}\right)
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(b) Bayes' rule estimator with squared error loss

Bayes' rule estimator with squared error loss is posterior mean. Note that the mean of $\mathrm{Gamma}(\alpha, \beta)$ is $\alpha\beta$.

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Bayes' rule estimator with squared error loss is posterior mean. Note that the mean of $\mathrm{Gamma}(\alpha, \beta)$ is $\alpha\beta$.

$$\begin{aligned}
\pi(\theta|\mathbf{x}) &= \mathrm{Gamma}\left(\alpha + n - 1, \frac{1}{\beta^{-1} + \sum_{i=1}^{n} x_i}\right) \\
E[\theta|\mathbf{x}] &= E[\pi(\theta|\mathbf{x})] \\
&= \frac{\alpha + n - 1}{\beta^{-1} + \sum_{i=1}^{n} x_i}
\end{aligned}$$

Summary

Today

- E-M Algorithm
- Practice Problems for the Final Exam

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- E-M Algorithm
- Practice Problems for the Final Exam

Next Lectures

- Bayesian Tests
- Bayesian Intervals
- More practice problems