Biostatistics 615/815 Lecture 8: Trees and Hash Tables

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Average time complexity of container operations

	SEARCH	INSERT	DELETE
Array	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
SortedArray	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n)$
List	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
Hash	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Recap - List implementation : class myListNode

myListNode.h

```
// myListNode class is only accessible from myList class
template<class T>
class myListNode {
protected:
  T value;          // the value of each element
  myListNode<T>* next;  // pointer to the next element
  myListNode(T v, myListNode<T>* n) : value(v), next(n) {} // constructor
  ~myListNode();
  int search(T x, int curPos);
  myListNode<T>* remove(T x, myListNode<T>*& prevNext);
  template <class S> friend class myList; // allow full access to myList class
};
```

Recap - Removing an element from a list

myListNode.h

```
template <class T>
// pass the pointer to [prevElement->next] so that we can change it
myListNode<T>* myListNode<T>::remove(T x, myListNode<T>*& prevNext) {
  if ( value == x ) {   // if FOUND
    prevNext = next;    // *pPrevNext was this, but change to next
    next = NULL;        // disconnect the current object from the list
    return this;        // and return it so that it can be destroyed
  }
  else if ( next == NULL ) {
    return NULL;        // return NULL if NOT_FOUND
  }
  else {
    return next->remove(x, next); // recursively call on the next element
  }
}
```

Recap - Implementation of binary search tree

myTree.h

```
template <class T>
class myTree {
protected:
  myTreeNode<T>* pRoot;       // tree contains pointer to root
  myTree(myTree& a) {};       // prevent copying
public:
  myTree() : pRoot(NULL) {} // initially root is empty
  ~myTree() { if ( pRoot != NULL ) delete pRoot; } // destructor
  void insert(T x);
  int search(T x);
  bool remove(T x);
};
```

Recap - Implementation of binary search tree

myTreeNode.h

```
template <class T>
class myTreeNode {
  T value;   // key value
  int size;  // total number of nodes in the subtree
  myTreeNode<T>* left;  // pointer to the left subtree
  myTreeNode<T>* right; // pointer to the right subtree
  myTreeNode(T x, myTreeNode<T>* l, myTreeNode<T>* r); // constructors
  ~myTreeNode();            // destructors
  void insert(T x); // insert an element
  int search(T x);
  myTreeNode<T>* remove(T x, myTreeNode<T>** ppSelf);
  T getMax();               // maximum value in the subtree
  T getMin();               // minimum value in the subtree
};
```

Binary search tree : Remove

myTreeNode.h

```
template <class T>
myTreeNode<T>* myTreeNode<T>::remove(T x, myTreeNode<T>*& pSelf) {
  if ( x == value ) {  // key was found
    if ( ( left == NULL ) && ( right == NULL ) ) { // no child
      pSelf = NULL;  // the parent has no offspring any more
      return this;
    }
    else if ( left == NULL ) { // only left is NULL
      pSelf = right; // right becomes the new offspring
      right = NULL;  // isolate the node so that deletion won't propagate
      return this;
    }
    else if ( right == NULL ) {  // only right is NULL
      pSelf = left;  // left node becomes the new offsprinf
      left = NULL;   // isolate the node so that deletion won't propagate
      return this;
    } // ....
```

Binary search tree : REMOVE (cont'd)

myTreeNode.h

```
    else { // neither left nor right is NULL
      // choose which subtree to delete
      myTreeNode<T>* p;
      if ( left->size > right->size ) { // if left subtree is larger
        T m = left->getMax();       // copy the largest value among them
        p = left->remove(m, left);  // to current node, and delete the node
        value = m;
      }
      else {
        T m = right->getMin();       // copy smallest value among them
        p = right->remove(m, right); // to current node, and delete the node
        value = m;
      }
      return p;
    }
  }
  // ....
```

Binary search tree : REMOVE (cont'd)

myTreeNode.h

```
  else if ( x < value ) {
    if ( left == NULL )
      return NULL;
    else
      return left->remove(x, left);
  }
  else { // x > value
    if ( right == NULL )
      return NULL;
    else
      return right->remove(x, right);
  }
}
```

Binary search tree : GETMAX and GETMIN

myTreeNode.h

```
template <class T>
T myTreeNode<T>::getMax() {  // return the largest value
  if ( right == NULL ) return value;
  else return right->getMax();
}

template <class T>
T myTreeNode<T>::getMin() {  // return the smallest value
  if ( left == NULL ) return value;
  else return left->getMin();
}
```

If you want to print a tree...

myTreeNode.h

```
template <class T> void myTreeNode<T>::print() {
  std::cout << "[ ";
  if ( left != NULL ) left->print();
  else std::cout << "[ NULL ]";
  std::cout << " , (" << value << "," << size << ") , ";
  if ( right != NULL ) right->print();
  else std::cout << "[ NULL ]";
  std::cout << " ]";
}
```

myTree.h

```
template <class T> void myTree<T>::print() {
  if ( pRoot != NULL ) pRoot->print();
  else std::cout << "(EMPTY TREE)";
  std::cout << std::endl;
}
```

Summary - Binary Search Tree

- Key Features
  - Fast insertion, search, and removal
  - Implementation is much more complicated
- Class Structure
  - `myTree` class to keep the root node
  - `myTreeNode` class to store key and up to two children
- Key Algorithms

  Insert : Traverse the tree in sorted order and create a new node in the first leaf node.

  Search : Divide-and-conquer algorithms

  Remove : Move the nearest leaf element among the subtree and destroy it.

Two types of containers

Containers for single-valued objects - last lecture

- $\text{INSERT}(T, x)$ - Insert x to the container.
- $\text{SEARCH}(T, x)$ - Returns the location/index/existence of x.
- $\text{REMOVE}(T, x)$ - Delete x from the container if exists
- STL examples include `std::vector`, `std::list`, `std::deque`, `std::set`, and `std::multiset`.

Containers for `(key,value)` pairs - this lecture

- $\text{INSERT}(T, x)$ - Insert $(x.key, x.value)$ to the container.
- $\text{SEARCH}(T, k)$ - Returns the value associated with key k.
- $\text{REMOVE}(T, x)$ - Delete element x from the container if exitst
- Examples include `std::map`, `std::multimap`, and `__gnu_cxx::hash_map`

Direct address tables

An example (key,value) container

- $U = \{0, 1, \cdots, N-1\}$ is possible values of keys (N is not huge)
- No two elements have the same key

Direct address table : a constant-time continaer

Let $T[0, \cdots, N-1]$ be an array space that can contain N objects

- $\textsc{Insert}(T, x)$: $T[x.key] = x$
- $\textsc{Search}(T, k)$: $\textsc{return}\ T[k]$
- $\textsc{Remove}(T, x)$: $T[x.key] = \textsc{Nil}$

Analysis of direct address tables

Time complexity

- Requires a single memory access for each operation
- $O(1)$ - constant time complexity

Memory requirement

- Requires to pre-allocate memory space for any possible input value
- $2^{32} = 4GB\times$(size of data) for 4 bytes (32 bit) key
- $2^{64} = 18EB(1.8 \times 10^7 TB)\times$(size of data) for 8 bytes (64 bit) key
- An infinite amount of memory space needed for storing a set of arbitrary-length strings (or exponential to the length of the string)

Hash Tables

Key features

- $O(1)$ complexity for INSERT, SEARCH, and REMOVE
- Requires large memory space than the actual content for maintainng good performance
- But uses much smaller memory than direct-addres tables

Key components

- Hash function
  - $h(x.key)$ mapping key onto smaller 'addressible' space H
  - Total required memory is the possible number of hash values
  - Good hash function minimize the possibility of key collisions
- Collision-resolution strategy, when $h(k_1) = h(k_2)$.

Chained hash : A simple example

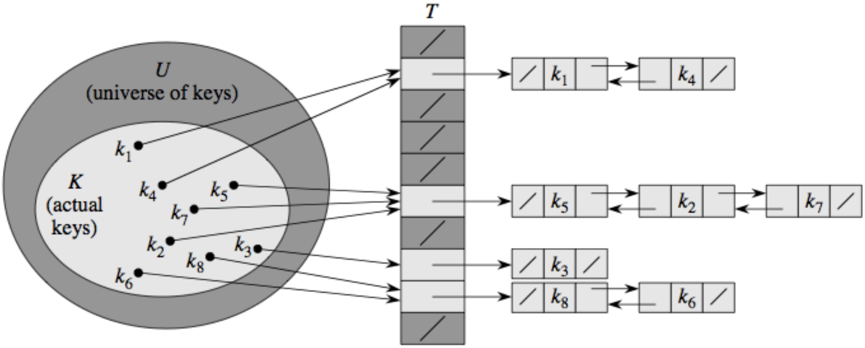
A good hash function

- Assume that we have a good hash function $h(x.key)$ that 'fairly uniformly' distribute key values to H
- What makes a good hash function will be discussed later today.

A `ChainedHash`

- Each possible hash key contains a linked list
- Each linked list is originally empty
- An input `(key,value)` pair is appened to the linked list when inserted
- $O(1)$ time complexity is guaranteed when no collision occurs
- When collision occurs, the time complexity is proportional to size of linked list assocated with $h(x.key)$

Illustration of ChainedHash



Simplfied algorithms on CHAINEDHASH

INITIALIZE(T)

- Allocate an array of list of size m as the number of possible key values

INSERT(T, x)

- Insert x at the head of list $T[h(x.key)]$.

SEARCH(T, k)

- Search for an element with key k in list $T[h(k)]$.

REMOVE(T, x)

- Delete x fom the list $T[h(x.key)]$.

Analysis of hashing with chaining

Assumptions

- Simple uniform hashing
  - $\Pr(h(k_1) = h(k_2)) = 1/m$ input key pairs k_1 and k_2.
- n is the number of elements stores
- Load factor $\alpha = n/m$.

Expected time complexity for SEARCH

- $X_{ij} \in \{0, 1\}$ a random variable of key collision between x_i and x_j.
- $E[X_{ij}] = 1/m$.

$$T(n) = \frac{1}{n} E\left[\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n}(X_{ij})\right)\right] = \Theta(1+\alpha)$$

Interesting properties (under uniform hash)

Probability of an empty slot

$$\Pr(k_1 \neq k, k_2 \neq k, \cdots, k_n \neq k) = \left(1 - \frac{1}{m}\right)^n \approx e^{-\alpha}$$

Birthday paradox : expected # of elements before the first collision

$$Q(H) \approx \sqrt{\frac{\pi}{2} m}$$

Coupon collector problem : expect # of elements to fill every slot

$$\sum_{i=1}^{m} \frac{m}{i} \approx m(\ln m + 0.577)$$

Hash functions

Making a good hash functions

- A hash function $h(k)$ is a deterministic function from $k \in K$ onto $h(k) \in H$.
- A good hash function distributes map the keys to hash values as uniform as possible
- The uniformity of hash function should not be affected by the pattern of input sequences

Example hash functions

- $k \in [0, 1)$, $h(k) = \lfloor km \rfloor$
- $k \in \mathbb{N}$, $h(k) = k \mod m$

'Good' and 'bad' hash functions

An example : $h(k) = \lfloor km \rfloor$

- When the input if uniformly distributed
  - $h(k)$ is uniformly distributed between 0 and $m-1$.
  - $h(k)$ is a good hash function
- When the input is skewed : $\Pr(k < 1/m) = 0.9$
  - More than 80% of input key pairs will have collisions
  - $h(k)$ is a bad hash function
  - Time complexity is close to a single linked list

Good hash functions

- 'Goodness' of a hash function can be dependent on the data
- If it is hard to create adversary inputs to make the hash function 'bad', it is generally a good hash function.

Examples of good hash functions

For integers

- Make the hash size m to be a large prime
- $h(k) = k \mod m$

For floating point values $k \in [0, 1)$

- Make the hash size m to be a large prime
- $h(k) = \lfloor k * N \rfloor \mod m$ for a large number N.

For strings

- Pretend the string is a number with numeral system of $|\Sigma|$, where Σ is the set of possible characters.
- Apply the same hash function for integers

Open Addressing

Chained Hash - Pros and Cons

- △ Easy to understand
- △ Behavior at collision is easy to track
- ▽ Every slots maintains pointer - extra memory consumption
- ▽ Inefficient to dereference pointers for each access
- ▽ Larger and unpredictable memory consumption

Open Addressing

- Store all the elements within an array
- Resolve conflicts based on predefined probing rule.
- Avoid using pointers - faster and more memory efficient.
- Implementation of REMOVE can be very complicated

Probing in open hash

Modified hash functions

- $h : K \times H \rightarrow H$
- For every $k \in K$, the probe sequence $< h(k,0), h(k,1), \cdots, h(k, m-1) >$ must be a permutation of $< 0, 1, \cdots, m-1 >$.

Algorithm OpenHashInsert

Data: T : hash, k : key value to insert
Result: k is inserted to T

for $i = 0$ **to** $m - 1$ **do**
    $j = h(k, i)$ **if** $T[j] ==$ Nil **then**
        $T[j] = k$;
        return j;
    end
end
error *"hash table overflow"*;

Algorithm OpenHashSearch

Data: T : hash, k : key value to search
Result: Return $T[k]$ if exist, otherwise return NIL

```
for i = 0 to m − 1 do
    j = h(k, i);
    if T[j] == k then
        return j;
    end
    else if T[j] == NIL then
        return NIL;
    end
end
return NIL;
```

Strategies for Probing

Linear Probing

- $h(k, i) = (h'(k) + i) \mod m$
- Easy to implement
- Suffer from primary clustering, increasing the average search time

Quadratic Probing

- $h(k, i) = (h'(k) + c_1 i + c_2 i^2) \mod m$
- Beter than linear probing
- Seconary clustering : $h(k_1, 0) = h(k_2, 0)$ implies $h(k_1, i) = k(k_2, i)$

Strategies for Probing

Double Hashing

- $h(k, i) = (h_1(k) + ih_2(k)) \mod m$
- The probe sequence depends in two ways upon k.
- For example, $h_1(k) = k \mod m$, $h_2(k) = 1 + (k \mod m')$
- Avoid clustering problem
- Performance close to ideal scheme of uniform hashing.

Hash tables : summary

- Linear-time performance container with larger storage
- Key components
  - Hash function
  - Conflict-resolution strategy
- Chained hash
  - Linked list for every possible key values
  - Large memory consumption + deferencing overhead
- Open Addressing
  - Probing strategy is important
  - Double hashing is close to ideal hashing

When are binary search trees better than hash tables?

- When the memory efficiency is more important than the search efficiency
- When many input key values are not unique
- When querying by ranges or trying to find closest value.

Recap: Divide and conquer algorithms

Good examples of divide and conquer algorithms

- TOWEROFHANOI
- MERGESORT
- QUICKSORT
- BINARYSEARCHTREE algorithms

These algorithms divide a problem into smaller and disjoint subproblems until they become trivial.

A divide-and-conquer algorithms for Fibonacci numbers

Fibonacci numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & n > 1 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases}$$

A recursive implementation of fibonacci numbers

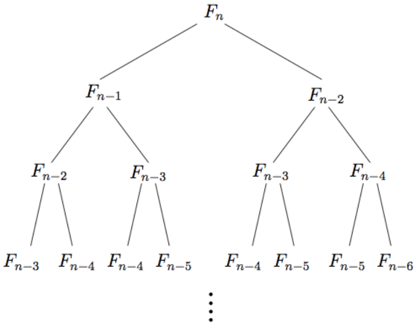
```
int fibonacci(int n) {
  if ( n < 2 ) return n;
  else return fibonacci(n-1)+fibonacci(n-2);
}
```

Performance of recursive FIBONACCI

Computational time

- 4.4 seconds for calculating F_{40}
- 49 seconds for calculating F_{45}
- ∞ seconds for calculating F_{100}!

What is happening in the recursive FIBONACCI



Time complexity of redundant FIBONACCI

$$
\begin{aligned}
T(n) &= T(n-1) + T(n-2) \\
T(1) &= 1 \\
T(0) &= 1 \\
T(n) &= F_{n+1}
\end{aligned}
$$

The time complexity is exponential

A non-redundant FIBONACCI

```
int fibonacci(int n) {
  int* fibs = new int[n+1];
  fibs[0] = 0;
  fibs[1] = 1;
  for(int i=2; i <= n; ++i) {
    fibs[i] = fibs[i-1]+fibs[i-2];
  }
  int ret = fibs[n];
  delete [] fibs;
  return ret;
}
```

Key idea in non-redundant FIBONACCI

- Each F_n will be reused to calculate F_{n+1} and F_{n+2}
- Store F_n into an array so that we don't have to recalculate it

A recursive, but non-redundant FIBONACCI

```
int fibonacci(int* fibs, int n) {
  if ( fibs[n] > 0 ) {
    return fibs[n];    // reuse stored solution if available
  }
  else if ( n < 2 ) {
    return n;          // terminal condition
  }
  fibs[n] = fibonacci(n-1) + fibonacci(n-2); // store the solution once computed
  return fibs[n];
}
```

Summary

Today

- Tree
- Hash Table
- Dynamic programming

Next Lecture

- More on dynamic programming
- Graph algorithms

Reading materials

- CLRS Chapter 15