Biostatistics 602 - Statistical Inference
Lecture 07
Exponential Family

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January 31st, 2013

Last Lecture

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3. What is the characteristic shared among non-constant functions of complete statistics?
4. What is the Basu's Theorem?
5. Any example where Basu's Theorem is helpful?

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Curved and Full Exponential Families

Definition

For an exponential family, if $d = \dim(\boldsymbol{\theta}) < k$, then this exponential family is called *curved exponential family*. if $d = \dim(\boldsymbol{\theta}) = k$, then this exponential family is called *full exponential family*.

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- $\mathcal{N}(\mu, \mu), \mu \in \mathbb{R}$ is also a curved exponential family

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where $c^*(\boldsymbol{\eta}) = c \circ w(\boldsymbol{\theta})$. This alternative parametrization is most often used in a GLM (Generalized Linear Model) course.

Sufficient Statistic for Exponential Families

Theorem 6.2.10

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} f_X(x|\boldsymbol{\theta})$ that belongs to an exponential family given by

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$$T(\mathbf{X}) = \left(\sum_{j=1}^{n} t_1(X_j), \cdots, \sum_{j=1}^{n} t_k(X_j)\right)$$

Sufficient Statistic for Normal Distribution

Problem

Let $X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$, and σ^2 is known. Find a sufficient statistic for μ.

Sufficient Statistic for Normal Distribution - Solution

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where

$$
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Therefore, $T(\mathbf{X}) = \sum_{i=1}^n t(X_i) = \sum_{i=1}^n X_i$ is a sufficient statistic for μ by Theorem 6.2.10.

Theorem 5.2.11

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If the set $\{w_1(\boldsymbol{\theta}), \cdots, w_k(\boldsymbol{\theta}),\ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ contains an open subset of $\mathbb{R}^k$, then the distribution of $\mathbf{T}(\mathbf{X})$ is an exponential family of the form

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$$f_X(x|\boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta}) \exp\left[\sum_{j=1}^{k} w_j(\boldsymbol{\theta})t_j(x)\right]$$

is a member of an exponential family. Define a statistic $T(\mathbf{X})$ by

$$\mathbf{T}(\mathbf{X}) = \left(\sum_{j=1}^{n} t_1(X_j), \cdots, \sum_{j=1}^{n} t_k(X_j)\right)$$

If the set $\{w_1(\boldsymbol{\theta}), \cdots, w_k(\boldsymbol{\theta}),\ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}\}$ contains an open subset of $\mathbb{R}^k$, then the distribution of $\mathbf{T}(\mathbf{X})$ is an exponential family of the form

$$f_T(u_1, \cdots, u_k|\boldsymbol{\theta}) = H(u_1, \cdots, u_k)[c(\boldsymbol{\theta})]^n \exp\left[\sum_{j=1}^{k} w_j(\boldsymbol{\theta})u_i\right]$$

Theorem 6.2.25

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is complete as long as the parameter space $\boldsymbol{\Theta}$ contains an open set in $\mathbb{R}^k$

What is the "open set"?

Definition : Open Set

A set A is open in $\mathbb{R}^k$ of for every $x \in A$, there exists a ϵ-ball $B(x, \epsilon)$ around x such that $B(x, \epsilon) \subset A$.

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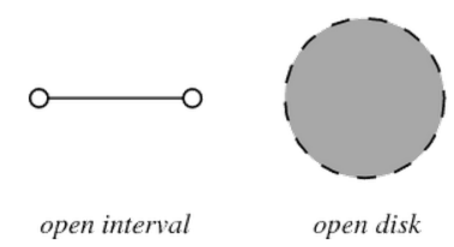
Examples (from Wolfram MathWorld)



Examples of open set test

- $A = (-1, 1)$: A is open in $\mathbb{R}$

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- $A = \{(x, y) : x^2 + y^2 < 1\}$: A is open in $\mathbb{R}^2$

Exponential Family Example

Problem

$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} f_X(x|\theta) = \theta x^{\theta-1}$ where $0 < x < 1, \theta > 0$. Is $\prod_{i=1}^n X_i$ (1) a sufficient statistic? (2) a complete statistic? (3) a minimal sufficient statistic?

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- Show that $f_X(x|\theta)$ is a member of an exponential family.
- Apply Theorem 6.2.10 to obtain a sufficient statistic and see if it is equivalent to or related to $\prod_{i=1}^n X_i$.
- Apply Theorem 6.2.25 to show that it is complete.
- If they are both sufficient and complete, Theorem 6.2.28 will imply that it is also a minimal sufficient statistic.

$f_X(x|\theta)$ belong to an exponential family

$$f_X(x|\theta) = \theta x^{\theta-1} I(0 < x < 1)$$

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\begin{aligned}
f_X(x|\theta) &= \theta x^{\theta-1} I(0 < x < 1) \\
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where

$$\begin{cases} h(x) = I(0 < x < 1) x^{-1} \\ \\ \\ \end{cases}$$

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\begin{cases}
h(x) = I(0 < x < 1) x^{-1} \\
c(\theta) = \theta \\
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Therefore, $f_X(x|\theta)$ belongs to an exponential family.

Apply Theorem 6.2.10

$$f_X(x|\theta) = h(x)c(\theta)\exp(w(\theta)t(x))$$

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Apply Theorem 6.2.10

$$f_X(x|\theta) \quad = \quad h(x)\,c(\theta)\exp(w(\theta)\,t(x))$$

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By Theorem 6.2.10, $T(\mathbf{X}) = \sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} \log X_i$ is a sufficient statistic for θ.

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$$\prod_{i=1}^n X_i = \exp\left(\log \prod_{i=1}^n X_i\right) = \exp\left(\sum_{i=1}^n \log X_i\right) = e^{T(\mathbf{X})}$$

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$$\prod_{i=1}^n X_i = \exp\left(\log \prod_{i=1}^n X_i\right) = \exp\left(\sum_{i=1}^n \log X_i\right) = e^{T(\mathbf{X})}$$

Because $\prod_{i=1}^n X_i$ is an one-to-one function of $T(\mathbf{X})$, it is also a sufficient statistic.

Apply Theorem 6.2.25 and 6.2.28

$T(\mathbf{X})$ is a complete statistic

Let $A = \{w(\theta) : \theta \in \Omega\} = \{\theta : \theta > 0\}$. A contains an open subset in $\mathbb{R}$.

Apply Theorem 6.2.25 and 6.2.28

$T(\mathbf{X})$ is a complete statistic

Let $A = \{w(\theta) : \theta \in \Omega\} = \{\theta : \theta > 0\}$. A contains an open subset in $\mathbb{R}$. By Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^{n} \log X_i$ is a complete statistic for θ.

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Let $A = \{w(\theta) : \theta \in \Omega\} = \{\theta : \theta > 0\}$. A contains an open subset in $\mathbb{R}$. By Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^{n} \log X_i$ is a complete statistic for θ.

$T(\mathbf{X})$ is a minimal sufficient statistic

By Theorem 6.2.28, because $T(\mathbf{X})$ is both sufficient and complete, it is also minimal sufficient.

Apply Theorem 6.2.25 and 6.2.28

$T(\mathbf{X})$ is a complete statistic

Let $A = \{w(\theta) : \theta \in \Omega\} = \{\theta : \theta > 0\}$. A contains an open subset in $\mathbb{R}$. By Theorem 6.2.25, $T(\mathbf{X}) = \sum_{i=1}^n \log X_i$ is a complete statistic for θ.

$T(\mathbf{X})$ is a minimal sufficient statistic

By Theorem 6.2.28, because $T(\mathbf{X})$ is both sufficient and complete, it is also minimal sufficient.

$\prod_{i=1}^n X_i = e^{T(\mathbf{X})}$ is also minimal sufficient and complete

Because $\prod_{i=1}^n X_i = e^{T(\mathbf{X})}$ is an one-to-one function of $T(\mathbf{X})$, $\prod_{i=1}^n X_i$ is sufficient, complete, and minimal sufficient.

Exponential Family

Today

- Curved and full exponential families
- Sufficient statistics for exponential families
- Complete statistics for exponential families

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Next Lecture

- Review of Chapter 6