Biostatistics 615/815 Lecture 8: Hash Tables, and Dynamic Programming

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Announcements

Homework #2

- For problem 3, assume that all the input values are unique
- Include the class definition into `myTree.h` and `myTreeNode.h` (do not make `.cpp` file)
- The homework `.tex` file containing the source code is uploaded in the class web page

815 projects

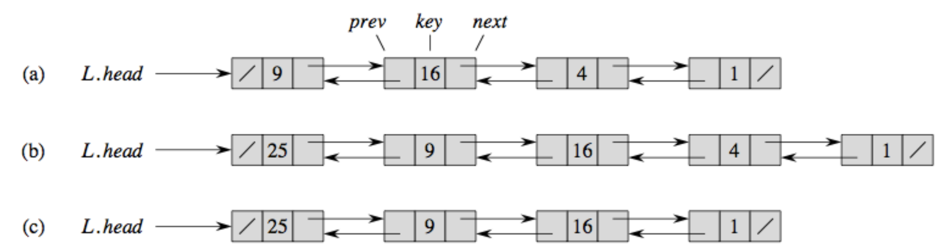
- Instructor sent out E-mails to individually today morning

Recap : Elementary data structures

	SEARCH	INSERT	REMOVE
Array	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
SortedArray	$\Theta(\log n)$	$\Theta(n)$	$\Theta(n)$
List	$\Theta(n)$	$\Theta(1)$	$\Theta(n)$
Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
Hash	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

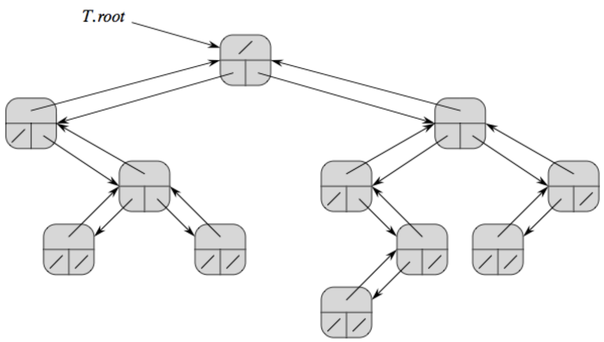
- Array or list is simple and fast enough for small-sized data
- Tree is easier to scale up to moderate to large-sized data
- Hash is the most robust for very large datasets

Recap: Example of a linked list



- Example of a doubly-linked list
- Singly-linked list if `prev` field does not exist

Recap: An example binary search tree



- Pointers to left and right children (NIL if absent)
- Pointers to its parent can be omitted.

Correction: Building your program (lecture 6)

Individually compile and link - Does NOT work with template

- Include the content of your `.cpp` files into `.h`
- For example, `Main.cpp` includes `myArray.h`

```
user@host:~/> g++ -o myArrayTest Main.cpp
```

Or create a Makefile and just type 'make'

```
all: myArrayTest  # binary name is myArrayTest

myArrayTest: Main.cpp  # link two object files to build binary
    g++ -o myArrayTest Main.cpp  # must start with a tab

clean:
    rm *.o myArrayTest
```


Today

Data structure

- Hash table

Dynamic programming

- Divide and conquer vs dynammic programming

Two types of containers

Containers for single-valued objects - last lectures

- INSERT(T, x) - Insert x to the container.
- SEARCH(T, x) - Returns the location/index/existence of x.
- REMOVE(T, x) - Delete x from the container if exists
- STL examples include `std::vector`, `std::list`, `std::deque`, `std::set`, and `std::multiset`.

Containers for `(key,value)` pairs - this lecture

- INSERT(T, x) - Insert $(x.key, x.value)$ to the container.
- SEARCH(T, k) - Returns the value associated with key k.
- REMOVE(T, x) - Delete element x from the container if existst
- Examples include `std::map`, `std::multimap`, and `__gnu_cxx::hash_map`

Direct address tables

An example `(key,value)` container

- $U = \{0, 1, \cdots, N-1\}$ is possible values of keys (N is not huge)
- No two elements have the same key

Direct address table : a constant-time continaer

Let $T[0, \cdots, N-1]$ be an array space that can contain N objects

- $\text{INSERT}(T, x)$: $T[x.key] = x$
- $\text{SEARCH}(T, k)$: RETURN $T[k]$
- $\text{REMOVE}(T, x)$: $T[x.key] = \text{NIL}$

Analysis of direct address tables

Time complexity

- Requires a single memory access for each operation
- $O(1)$ - constant time complexity

Memory requirement

- Requires to pre-allocate memory space for any possible input value
- $2^{32} = 4GB\times$(size of data) for 4 bytes (32 bit) key
- $2^{64} = 18EB(1.8 \times 10^7 TB)\times$(size of data) for 8 bytes (64 bit) key
- An infinite amount of memory space needed for storing a set of arbitrary-length strings (or exponential to the length of the string)

Hash Tables

Key features

- $O(1)$ complexity for INSERT, SEARCH, and REMOVE
- Requires large memory space than the actual content for maintainng good performance
- But uses much smaller memory than direct-addres tables

Key components

- Hash function
  - $h(x.key)$ mapping key onto smaller 'addressible' space H
  - Total required memory is the possible number of hash values
  - Good hash function minimize the possibility of key collisions
- Collision-resolution strategy, when $h(k_1) = h(k_2)$.

Chained hash : A simple example

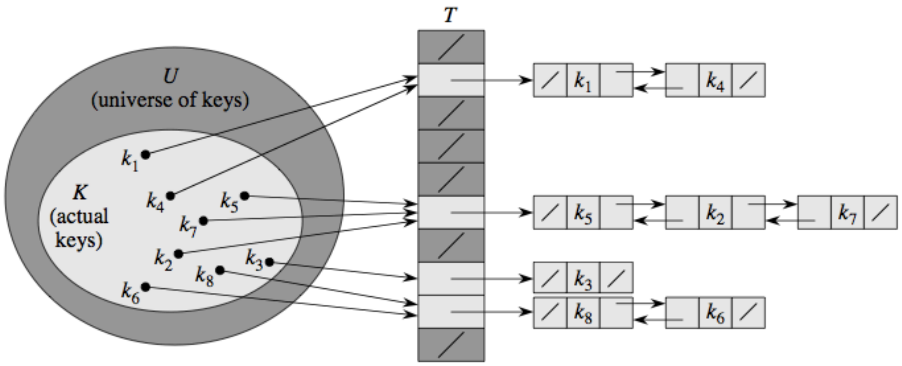
A good hash function

- Assume that we have a good hash function $h(x.key)$ that 'fairly uniformly' distribute key values to H
- What makes a good hash function will be discussed later today.

A `ChainedHash`

- Each possible hash key contains a linked list
- Each linked list is originally empty
- An input `(key,value)` pair is appened to the linked list when inserted
- $O(1)$ time complexity is guaranteed when no collision occurs
- When collision occurs, the time complexity is proportional to size of linked list assocated with $h(x.key)$

Illustration of ChainedHash



Simplfied algorithms on ChainedHash

Initialize(T)

- Allocate an array of list of size m as the number of possible key values

Insert(T, x)

- Insert x at the head of list $T[h(x.key)]$.

Search(T, k)

- Search for an element with key k in list $T[h(k)]$.

Remove(T, x)

- Delete x fom the list $T[h(x.key)]$.

Analysis of hashing with chaining

Assumptions

- Simple uniform hashing
  - $\Pr(h(k_1) = h(k_2)) = 1/m$ input key pairs k_1 and k_2.
- n is the number of elements stores
- Load factor $\alpha = n/m$.

Expected time complexity for Search

- $X_{ij} \in \{0, 1\}$ a random variable of key collision between x_i and x_j.
- $E[X_{ij}] = 1/m$.

$$T(n) = \frac{1}{n} E\left[\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n}(X_{ij})\right)\right] = \Theta(1+\alpha)$$

Interesting properties (under uniform hash)

Probability of an empty slot

$$\Pr(k_1 \neq k, k_2 \neq k, \cdots, k_n \neq k) = \left(1 - \frac{1}{m}\right)^n \approx e^{-\alpha}$$

Birthday paradox : expected # of elements before the first collision

$$Q(H) \approx \sqrt{\frac{\pi}{2} m}$$

Coupon collector problem : expect # of elements to fill every slot

$$\sum_{i=1}^{m} \frac{m}{i} \approx m(\ln m + 0.577)$$

Hash functions

Making a good hash functions

- A hash function $h(k)$ is a deterministic function from $k \in K$ onto $h(k) \in H$.
- A good hash function distributes map the keys to hash values as uniform as possible
- The uniformity of hash function should not be affected by the pattern of input sequences

Example hash functions

- $k \in [0, 1)$, $h(k) = \lfloor km \rfloor$
- $k \in \mathbb{N}$, $h(k) = k \mod m$

'Good' and 'bad' hash functions

An example : $h(k) = \lfloor km \rfloor$

- When the input if uniformly distributed
  - $h(k)$ is uniformly distributed between 0 and $m - 1$.
  - $h(k)$ is a good hash function
- When the input is skewed : $\Pr(k < 1/m) = 0.9$
  - More than 80% of input key pairs will have collisions
  - $h(k)$ is a bad hash function
  - Time complexity is close to a single linked list

Good hash functions

- 'Goodness' of a hash function can be dependent on the data
- If it is hard to create adversary inputs to make the hash function 'bad', it is generally a good hash function.

Examples of good hash functions

For integers

- Make the hash size m to be a large prime
- $h(k) = k \mod m$

For floating point values $k \in [0, 1)$

- Make the hash size m to be a large prime
- $h(k) = \lfloor k * N \rfloor \mod m$ for a large number N.

For strings

- Pretend the string is a number with numeral system of $|\Sigma|$, where Σ is the set of possible characters.
- Apply the same hash function for integers

Open Addressing

Chained Hash - Pros and Cons

- △ Easy to understand
- △ Behavior at collision is easy to track
- ▽ Every slots maintains pointer - extra memory consumption
- ▽ Inefficient to dereference pointers for each access
- ▽ Larger and unpredictable memory consumption

Open Addressing

- Store all the elements within an array
- Resolve conflicts based on predefined probing rule.
- Avoid using pointers - faster and more memory efficient.
- Implementation of REMOVE can be very complicated

Probing in open hash

Modified hash functions

- $h : K \times H \rightarrow H$
- For every $k \in K$, the probe sequence $< h(k,0), h(k,1), \cdots, h(k, m-1) >$ must be a permutation of $< 0, 1, \cdots, m-1 >$.

Algorithm OpenHashInsert

Data: T : hash, k : key value to insert
Result: k is inserted to T
for $i = 0$ **to** $m-1$ **do**
  $j = h(k,i)$ **if** $T[j]$ ==Nil **then**
    $T[j] = k$;
    return j;
  end
end
error *"hash table overflow"*;

Algorithm OpenHashSearch

Data: T : hash, k : key value to search
Result: Return $T[k]$ if exist, otherwise return Nil
for $i = 0$ **to** $m-1$ **do**
  $j = h(k,i)$;
  if $T[j] == k$ **then**
    return j;
  end
  else if $T[j]$ ==Nil **then**
    return Nil;
  end
end
return Nil;

Strategies for Probing

Linear Probing

- $h(k,i) = (h'(k) + i) \mod m$
- Easy to implement
- Suffer from primary clustering, increasing the average search time

Quadratic Probing

- $h(k,i) = (h'(k) + c_1 i + c_2 i^2) \mod m$
- Beter than linear probing
- Seconary clustering : $h(k_1, 0) = h(k_2, 0)$ implies $h(k_1, i) = k(k_2, i)$

Strategies for Probing

Double Hashing

- $h(k, i) = (h_1(k) + ih_2(k)) \mod m$
- The probe sequence depends in two ways upon k.
- For example, $h_1(k) = k \mod m$, $h_2(k) = 1 + (k \mod m')$
- Avoid clustering problem
- Performance close to ideal scheme of uniform hashing.

Hash tables : summary

- Linear-time performance container with larger storage
- Key components
  - Hash function
  - Conflict-resolution strategy
- Chained hash
  - Linked list for every possible key values
  - Large memory consumption + deferencing overhead
- Open Addressing
  - Probing strategy is important
  - Double hashing is close to ideal hashing

When are binary search trees better than hash tables?

- When the memory efficiency is more important than the search efficiency
- When many input key values are not unique
- When querying by ranges or trying to find closest value.

Recap: Divide and conquer algorithms

Good examples of divide and conquer algorithms

- TowerOfHanoi
- MergeSort
- QuickSort
- BinarySearchTree algorithms

These algorithms divide a problem into smaller and disjoint subproblems until they become trivial.

A divide-and-conquer algorithms for Fibonacci numbers

Fibonacci numbers

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & n > 1 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases}$$

A recursive implementation of fibonacci numbers

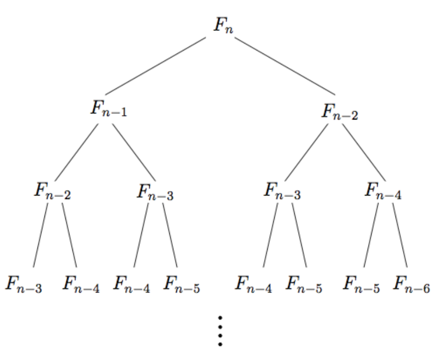
```
int fibonacci(int n) {
  if ( n < 2 ) return n;
  else return fibonacci(n-1)+fibonacci(n-2);
}
```


Performance of recursive FIBONACCI

Computational time

- 4.4 seconds for calculating F_{40}
- 49 seconds for calculating F_{45}
- ∞ seconds for calculating F_{100}!

What is happening in the recursive FIBONACCI



Time complexity of redundant FIBONACCI

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) \\ T(1) &= 1 \\ T(0) &= 1 \\ T(n) &= F_{n+1} \end{aligned}$$

The time complexity is exponential

A non-redundant FIBONACCI

```
int fibonacci(int n) {
  int* fibs = new int[n+1];
  fibs[0] = 0;
  fibs[1] = 1;
  for(int i=2; i <= n; ++i) {
    fibs[i] = fibs[i-1]+fibs[i-2];
  }
  int ret = fibs[n];
  delete [] fibs;
  return ret;
}
```


Key idea in non-redundant FIBONACCI

- Each F_n will be reused to calculate F_{n+1} and F_{n+2}
- Store F_n into an array so that we don't have to recalculate it

A recursive, but non-redundant FIBONACCI

```
int fibonacci(int* fibs, int n) {
  if ( fibs[n] > 0 ) {
    return fibs[n];    // reuse stored solution if available
  }
  else if ( n < 2 ) {
    return n;          // terminal condition
  }
  fibs[n] = fibonacci(n-1) + fibonacci(n-2); // store the solution once computed
  return fibs[n];
}
```


Summary

Today
- Hashing
- Dynamic programming

Next Lecture
- More on dynamic programming
- Graph algorithms

Reading materials
- CLRS Chapter 15