Biostatistics 602 - Statistical Inference
Lecture 18
Hypothesis Testing

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Last Lecture

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- What is the Asymptotic Relative Efficiency?
- What does mean that a statistic is asymptotically efficient?
- Is an MLE asymptotically efficient?

Asymptotic Normality

Definition: Asymptotic Normality

A statistic (or an estimator) $W_n(\mathbf{X})$ is *asymptotically normal* if

$$\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{\text{d}} \mathcal{N}(0, \nu(\theta))$$

for all θ

where $\xrightarrow{\text{d}}$ stands for "converge in distribution"

- $\tau(\theta)$: "asymptotic mean"
- $\nu(\theta)$: "asymptotic variance"

We denote $W_n \sim \mathcal{AN}\left(\tau(\theta), \frac{\nu(\theta)}{n}\right)$.

Central Limit Theorem

Central Limit Theorem

Assume $X_i \overset{\text{i.i.d.}}{\sim} f(x|\theta)$ with finite mean $\mu(\theta)$ and variance $\sigma^2(\theta)$.

$$\overline{X} \quad \sim \quad \mathcal{AN}\left(\mu(\theta), \frac{\sigma^2(\theta)}{n}\right)$$

$$\Leftrightarrow \sqrt{n}\left(\overline{X} - \mu(\theta)\right) \quad \xrightarrow{\text{d}} \quad \mathcal{N}(0, \sigma^2(\theta))$$

Theorem 5.5.17 - Slutsky's Theorem

If $X_n \xrightarrow{\text{d}} X$, $Y_n \xrightarrow{\text{P}} a$, where a is a constant,

1. $Y_n \cdot X_n \xrightarrow{\text{d}} aX$
2. $X_n + Y_n \xrightarrow{\text{d}} X + a$

Delta Method

Theorem 5.5.24 - Delta Method

Assume $W_n \sim \mathcal{AN}\left(\theta, \frac{\nu(\theta)}{n}\right)$. If a function g satisfies $g'(\theta) \neq 0$, then

$$g(W_n) \sim \mathcal{AN}\left(g(\theta), [g'(\theta)]^2 \frac{\nu(\theta)}{n}\right)$$

Asymptotic Efficiency

Definition : Asymptotic Efficiency for iid samples

A sequence of estimators W_n is asymptotically efficient for $\tau(\theta)$ if for all $\theta \in \Omega$,

$$\begin{aligned}
\sqrt{n}(W_n - \tau(\theta)) &\xrightarrow{\text{d}} \mathcal{N}\left(0, \frac{[\tau'(\theta)]^2}{I(\theta)}\right) \\
\iff W_n &\sim \mathcal{AN}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{nI(\theta)}\right) \\
I(\theta) &= E\left[\left\{\frac{\partial}{\partial\theta}\log f(X|\theta)\right\}^2 \middle| \theta\right] \\
&= -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta) \middle| \theta\right] \quad \text{(if interchangeability holds}
\end{aligned}$$

Note: $\frac{[\tau'(\theta)]^2}{nI(\theta)}$ is the C-R bound for unbiased estimators of $\tau(\theta)$.

Asymptotic Efficiency of MLEs

Theorem 10.1.12

Let $X_1, \cdots, X_n$ be iid samples from $f(x|\theta)$. Let $\hat{\theta}$ denote the MLE of θ. Under same regularity conditions, $\hat{\theta}$ is consistent and asymptotically normal for θ, i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \quad \xrightarrow{\mathrm{d}} \quad \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) \text{ for every } \theta \in \Omega$$

And if $\tau(\theta)$ is continuous and differentiable in θ, then

$$\sqrt{n}(\hat{\theta} - \theta) \quad \xrightarrow{\mathrm{d}} \quad \mathcal{N}\left(0, \frac{[\tau'(\theta)]}{I(\theta)}\right)$$

$$\Longrightarrow \tau(\hat{\theta}) \quad \sim \quad \mathcal{AN}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{nI(\theta)}\right)$$

Again, note that the asymptotic variance of $\tau(\hat{\theta})$ is Cramer-Rao lower bound for unbiased estimators of $\tau(\theta)$.

Hypothesis Testing

Definition

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- Alternative hypothesis : $H_1 : \theta \in \Omega_0^c$

$\theta \in \Omega = \Omega \cup \Omega^c$.

Simple and composite hypothesis

Simple hypothesis

Both H_0 and H_1 consist of only one parameter value.

- $H_0 : \theta = \theta_0 \in \Omega_0$
- $H_1 : \theta = \theta_1 \in \Omega_0^c$

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- One-sided hypothesis: $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$.
- Two-sided hypothesis: $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

An Example of Hypothesis

$$X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, 1)$$

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$$\begin{aligned} H_0 &: \theta = 0 \qquad \text{(no effect)} \\ H_1 &: \theta \neq 0 \qquad \text{(some effect)} \end{aligned}$$

Two-sided composite hypothesis.

Another Example of Hypothesis

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- We want to test whether the products produced by the machine is acceptable.

$$\begin{aligned} H_0 &: \theta \leq \theta_0 \qquad \text{(acceptable)} \\ H_1 &: \theta > \theta_0 \qquad \text{(unacceptable)} \end{aligned}$$

Hypothesis Testing Procedure

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Rejection region (R) on a hypothesis is usually defined through a test statistic $W(\mathbf{X})$. For example,

$$R_1 = \{\mathbf{x} : W(\mathbf{x}) > c, \mathbf{x} \in \mathcal{X}\}$$
$$R_2 = \{\mathbf{x} : W(\mathbf{x}) \leq c, \mathbf{x} \in \mathcal{X}\}$$

Example of hypothesis testing

$X_1, X_2, X_3 \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$. Consider hypothesis tests

$$\begin{aligned} H_0 &: \quad p \leq 0.5 \\ H_1 &: \quad p > 0.5 \end{aligned}$$

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- Test 1 : Reject H_0 if $\mathbf{x} \in \{(1,1,1)\}$
  $\iff$ rejection region $= \{(1,1,1)\}$
  $\iff$ rejection region $= \{\mathbf{x} : \sum x_i > 2\}$

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- Test 2 : Reject H_0 if $\mathbf{x} \in \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$
  $\Longleftrightarrow$ rejection region $= \{(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}$
  $\Longleftrightarrow$ rejection region $= \{\mathbf{x} : \sum x_i > 1\}$

Example

Let $X_1, \cdots, X_n$ be changes in blood pressure after a treatment.

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An example rejection region $R = \left\{ \mathbf{x} : \frac{\bar{x}}{s_{\mathbf{x}}/\sqrt{n}} > 3 \right\}$.

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Decision

Truth		Accept H_0	Reject H_0
	H_0	Correct Decision	Type I error
	H_1	Type II error	Correct Decision

Type I and Type II error

Type I error

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$$\Pr(\mathbf{X} \in R|\theta)$$

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Type II error

If $\theta \in \Omega_0^c$ (if the alternative hypothesis is true), the probability of making a type II error is

$$\Pr(\mathbf{X} \notin R|\theta) = 1 - \Pr(\mathbf{X} \in R|\theta)$$

Power function

Definition - The power function

The power function of a hypothesis test with rejection region R is the function of θ defined by

$$\beta(\theta) = \Pr(\mathbf{X} \in R|\theta) = \Pr(\text{reject } H_0|\theta)$$

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- Probability of type II error $= 1 - \beta(\theta)$ if $\theta \in \Omega_0^c$.

An ideal test should have power function satisfying $\beta(\theta) = 0$ for all $\theta \in \Omega_0$, $\beta(\theta) = 1$ for all $\theta \in \Omega_0^c$, which is typically not possible in practice.

Example of power function

Problem

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Test 1 rejects H_0 if and only if all "success" are observed. i.e.

$$\begin{aligned} R &= \{\mathbf{x} : \mathbf{x} = (1,1,1,1,1)\} \\ &= \{\mathbf{x} : \sum_{i=1}^{5} x_i = 5\} \end{aligned}$$

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2. What is the maximum probability of making type I error?
3. What is the probability of making type II error if $\theta = 2/3$?

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Type II error when $\theta = 2/3$

$$1 - \beta(\theta)|_{\theta=\frac{2}{3}} = 1 - \theta^5\Big|_{\theta=\frac{2}{3}} = 1 - (2/3)^5 = 211/243 \approx 0.868$$

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Test 2 rejects H_0 if and only if 3 or more "success" are observed. i.e.

$$R = \{\mathbf{x} : \sum_{i=1}^{5} x_i \geq 3\}$$

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1. Compute the power function
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Solution for Test 2

Power function

$$\beta(\theta) = \Pr(\sum X_i \geq 3|\theta) = \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{4}\theta^4(1-\theta) + \binom{5}{5}\theta^5$$

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We need to find the maximum of $\beta(\theta)$ for $\theta \in \Omega_0 = (0, 0.5]$

$$\beta'(\theta) = 30\theta^2(\theta-1)^2 > 0$$

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Maximum type I error

We need to find the maximum of $\beta(\theta)$ for $\theta \in \Omega_0 = (0, 0.5]$

$$\beta'(\theta) = 30\theta^2(\theta-1)^2 > 0$$

$\beta(\theta)$ is increasing in $\theta \in (0,1)$. Maximum type I error is $\beta(0.5) = 0.5$

Type II error when $\theta = 2/3$

$$1 - \beta(\theta)|_{\theta=\frac{2}{3}} = 1 - \theta^3(6\theta^2 - 15\theta + 10)\Big|_{\theta=\frac{2}{3}} \approx 0.21$$

Sizes and Levels of Tests

Size α test

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Level α test

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In other words, the maximum probability of making a type I error is equal or less than α.

Any size α test is also a level α test

Revisiting Previous Examples

Test 1

$$\sup_{\theta \in \Omega_0} \beta(\theta) = \sup_{\theta \in \Omega_0} \theta^5 = 0.5^5 = 0.03125$$

The size is 0.03125, and this is a level 0.05 test, or a level 0.1 test, but not a level 0.01 test.

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Test 2

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The size is 0.5

Constructing a good test

1. Construct all the level α test.

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2. Within this level of tests, we search for the test with Type II error probability as small as possible; equivalently, we want the test with the largest power if $\theta \in \Omega_0^c$.

Review on standard normal and t distribution

Quantile of standard normal distribution

Let $Z \sim \mathcal{N}(0,1)$ with pdf $f_Z(z)$ and cdf $F_Z(z)$. The α-th quantile z_α or $(1-\alpha)$-th quantile $z_{1-\alpha}$ of the standard distribution satisfy

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$$t_{n-1,1-\alpha} = -t_{n-1,\alpha}$$

Likelihood Ratio Tests (LRT)

Definition

Let $L(\theta|\mathbf{x})$ be the likelihood function of θ. The likelihood ratio test statistic for testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_0^c$ is

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$$\lambda(\mathbf{x}) = \frac{\sup_{\theta\in\Omega_0} L(\theta|\mathbf{x})}{\sup_{\theta\in\Omega} L(\theta|\mathbf{x})} = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}$$

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where $\hat{\theta}$ is the MLE of θ over $\theta \in \Omega$, and $\hat{\theta}_0$ is the MLE of θ over $\theta \in \Omega_0$ (restricted MLE).

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where $\hat{\theta}$ is the MLE of θ over $\theta \in \Omega$, and $\hat{\theta}_0$ is the MLE of θ over $\theta \in \Omega_0$ (restricted MLE).

The *likelihood ratio test* is a test that rejects H_0 if and only if $\lambda(\mathbf{x}) \leq c$ where $0 \leq c \leq 1$.

Properties of LRT

- For example
  - If $c = 1$, null hypothesis will always be rejected.
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$$\begin{aligned}\sup_{\theta \in \Omega_0} \Pr(\lambda(\mathbf{x}) \leq c) &= \sup_{\theta \in \Omega_0} \beta(\theta) \\ &= \sup_{\theta \in \Omega_0} \Pr(\text{reject } H_0) = \alpha\end{aligned}$$

Then we get a size α test.

Example of LRT

Problem

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For the LRT test and its power function

Solution

$$L(\theta|\mathbf{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i - \theta)^2}{2\sigma^2}\right]$$

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We need to find MLE of θ over $\Omega = (-\infty, \infty)$ and $\Omega_0 = (-\infty, \theta_0]$.

MLE of θ over $\Omega = (-\infty, \infty)$

To maximize $L(\theta|\mathbf{x})$, we need to maximize $\exp\left[-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2\sigma^2}\right]$, or equivalently to minimize $\sum_{i=1}^n (x_i - \theta)^2$.

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$$\begin{aligned}\sum_{i=1}^n (x_i - \theta)^2 &= \sum_{i=1}^n (x_i^2 + \theta^2 - 2\theta x_i) \\ &= n\theta^2 - 2\theta \sum_{i=1}^n x_i + \sum_{i=1}^n x_i^2\end{aligned}$$

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The equation above minimizes when $\theta = \hat{\theta} = \frac{\sum_{i=1}^n x_i}{n} = \overline{x}$.

MLE of θ over $\Omega_0 = (-\infty, \theta_0]$

- $L(\theta|\mathbf{x})$ is maximized at $\theta = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x}$ if $\overline{x} \leq \theta_0$.

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- However, if $\overline{x} \geq \theta_0$, $\overline{x}$ does not fall into a valid range of $\hat{\theta}_0$, and $\theta \leq \theta_0$, the likelihood function will be an increasing function. Therefore $\hat{\theta}_0 = \theta_0$.

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To summarize,

$$\hat{\theta}_0 = \begin{cases} \overline{X} & \text{if } \overline{X} \leq \theta_0 \\ \theta_0 & \text{if } \overline{X} > \theta_0 \end{cases}$$

Likelihood ratio test

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})} = \begin{cases} 1 & \text{if } \overline{X} \leq \theta_0 \\ \dfrac{\exp\left[-\frac{\sum_{i=1}^n (x_i-\theta_0)^2}{2\sigma^2}\right]}{\exp\left[-\frac{\sum_{i=1}^n (x_i-\overline{x})^2}{2\sigma^2}\right]} & \text{if } \overline{X} > \theta_0 \end{cases}$$

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$$= \begin{cases} 1 & \text{if } \overline{X} \leq \theta_0 \\ \exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] & \text{if } \overline{X} > \theta_0 \end{cases}$$

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$$= \begin{cases} 1 & \text{if } \overline{X} \leq \theta_0 \\ \exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] & \text{if } \overline{X} > \theta_0 \end{cases}$$

Therefore, the likelihood test rejects the null hypothesis if and only if

$$\exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] \leq c$$

and $\overline{x} \geq \theta_0$.

Specifying c

$$\exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] \quad \leq \quad c$$

Specifying c

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\begin{aligned}
\exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] &\leq c \\
\iff -\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2} &\leq \log c
\end{aligned}
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\exp\left[-\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2}\right] &\leq c \\
\Longleftrightarrow -\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2} &\leq \log c \\
\Longleftrightarrow (\overline{x}-\theta_0)^2 &\geq -\frac{2\sigma^2 \log c}{n}
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\iff -\frac{n(\overline{x}-\theta_0)^2}{2\sigma^2} &\leq \log c \\
\iff (\overline{x}-\theta_0)^2 &\geq -\frac{2\sigma^2 \log c}{n} \\
\iff \overline{x}-\theta_0 &\geq \sqrt{-\frac{2\sigma^2 \log c}{n}} \qquad (\because \overline{x} > \theta_0)
\end{aligned}
$$

Specifying c (cont'd)

So, LRT rejects H_0 if and only if

$$
\begin{aligned}
\overline{x} - \theta_0 &\geq \sqrt{-\frac{2\sigma^2 \log c}{n}} \\
\Longleftrightarrow \frac{\overline{x} - \theta_0}{\sigma/\sqrt{n}} &\geq \frac{\sqrt{-\frac{2\sigma^2 \log c}{n}}}{\sigma/\sqrt{n}} = c^*
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Specifying c (cont'd)

So, LRT rejects H_0 if and only if

$$\begin{aligned}\bar{x} - \theta_0 &\geq \sqrt{-\frac{2\sigma^2 \log c}{n}} \\ \Longleftrightarrow \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} &\geq \frac{\sqrt{-\frac{2\sigma^2 \log c}{n}}}{\sigma/\sqrt{n}} = c^*\end{aligned}$$

Therefore, the rejection region is

$$\left\{ \mathbf{x} : \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq c^* \right\}$$

Power function

$$\beta(\theta) \quad = \quad \Pr\left(\text{reject } H_0\right) = \Pr\left(\frac{\overline{X} - \theta_0}{\sigma/\sqrt{n}} \geq c^*\right)$$

Power function

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\begin{aligned}
\beta(\theta) &= \Pr\left(\text{reject } H_0\right) = \Pr\left(\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}} \geq c^*\right) \\
&= \Pr\left(\frac{\overline{X}-\theta+\theta-\theta_0}{\sigma/\sqrt{n}} \geq c^*\right)
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Since $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$, $\overline{X} \sim \mathcal{N}\left(\theta, \frac{\sigma^2}{n}\right)$. Therefore,

Power function

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$$\begin{aligned}\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} &\sim \mathcal{N}(0,1) \\ \Longrightarrow \beta(\theta) &= \Pr\left(Z \geq \frac{\theta_0-\theta}{\sigma/\sqrt{n}} + c^*\right)\end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$.

Making size α LRT

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Making size α LRT

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Making size α LRT

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\sup_{\theta \in \Omega_0} \beta(\theta) &= \alpha \\
\sup_{\theta \le \theta_0} \Pr\left(Z \ge \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^* \right) &= \alpha \\
\Pr\left(Z \ge c^* \right) &= \alpha
\end{aligned}$$

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\sup_{\theta \leq \theta_0} \Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right) &= \alpha \\
\Pr\left(Z \geq c^*\right) &= \alpha \\
c^* &= z_\alpha
\end{aligned}$$

Note that $\Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right)$ is maximized when θ is maximum (i.e. $\theta = \theta_0$).

Making size α LRT

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Note that $\Pr\left(Z \geq \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} + c^*\right)$ is maximized when θ is maximum (i.e. $\theta = \theta_0$).
Therefore, size α LRT test rejects H_0 if and only if $\frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq z_\alpha$.

Summary

Today

- Hypothesis Testing
- Likelihood Ratio Test

Summary

Today

- Hypothesis Testing
- Likelihood Ratio Test

Next Lecture

- More Hypothesis Testing