Biostatistics 615/815 Lecture 9: Dynamic Programming and Hidden Markov Models

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October 2nd, 2012

Minimum edit distance problem

Edit distance

Minimum number of letter insertions, deletions, substitutions required to transform one word into another

An example

FOOD → MOOD → MON^D → MONED → MONEY

Edit distance is 4 in the example above

More examples of edit distance

F	O	O		D
M	O	N	E	Y

A	L	G	O	R		I		T	H	M
A	L		T	R	U	I	S	T	I	C

- Similar representation to DNA sequence alignment
- Does the above alignment provides an optimal edit distance?

A dynamic programming solution

		A	L	G	O	R	I	T	H	M
	0	1	2	3	4	5	6	7	8	9
A	1	0	1	2	3	4	5	6	7	8
L	2	1	0	1	2	3	4	5	6	7
T	3	2	1	1	2	3	4	4	5	6
R	4	3	2	2	2	2	3	4	5	6
U	5	4	3	3	3	3	3	4	5	6
I	6	5	4	4	4	4	3	4	5	6
S	7	6	5	5	5	5	4	4	5	6
T	8	7	6	6	6	6	5	4	5	6
I	9	8	7	7	7	7	6	5	5	6
C	10	9	8	8	8	8	7	6	6	6

Recursively formulating the problem

- Input strings are $x[1, \cdots, m]$ and $y[1, \cdots, n]$.
- Let $x_i = x[1, \cdots, i]$ and $y_j = y[1, \cdots, j]$ be substrings of x and y.
- Edit distance $d(x, y)$ can be recursively defined as follows

$$d(x_i, y_j) = \begin{cases} i & j = 0 \\ j & i = 0 \\ \min \left\{ \begin{array}{l} d(x_{i-1}, y_j) + 1 \\ d(x_i, y_{j-1}) + 1 \\ d(x_{i-1}, y_{i-1}) + I(x[i] \neq y[j]) \end{array} \right\} & otherwise \end{cases}$$

- Similar to the Manhattan tourist problem, but with 3-way choice.
- Time complexity is $\Theta(mn)$.

Edit Distance Implementation

editDistance.cpp

```
#include <iostream>
#include <climits>
#include <string>
#include <vector>
#include "Matrix615.h"
int main(int argc, char** argv) {
  if ( argc != 3 ) {
    std::cerr << "Usage: editDistance [str1] [str2]" << std::endl;
    return -1;
  }
  std::string s1(argv[1]);
  std::string s2(argv[2]);
  Matrix615<int> cost(s1.size()+1, s2.size()+1, INT_MAX);
  Matrix615<int> move(s1.size()+1, s2.size()+1, -1);
  int optDist = editDistance(s1, s2, cost,move, cost.rowNums()-1, cost.colNums()-1);
  std::cout << "EditDistance is " << optDist << std::endl;
  printEdits(s1, s2, move);
  return 0;
}
```


editDistance() algorithm

editDistance.cpp

```
// note to declare the function before main()
int editDistance(std::string& s1, std::string& s2, Matrix615<int>& cost,
                 Matrix615<int>& move, int r, int c) {
  int iCost = 1, dCost = 1, mCost = 1; // insertion, deletion, mismatch cost

  if ( cost.data[r][c] == INT_MAX ) {
    if ( r == 0 && c == 0 ) { cost.data[r][c] = 0; }
    else if ( r == 0 ) {
      move.data[r][c] = 0;  // only insertion is possible
      cost.data[r][c] = editDistance(s1,s2,cost,move,r,c-1) + iCost;
    }
    else if ( c == 0 ) {
      move.data[r][c] = 1;  // only deletion is possible
      cost.data[r][c] = editDistance(s1,s2,cost,move,r-1,c) + dCost;
    }
```


editDistance() algorithm

editDistance.cpp

```
    else {  // compare 3 different possible moves and take the optimal one
      int iDist = editDistance(s1,s2,cost,move,r,c-1) + iCost;
      int dDist = editDistance(s1,s2,cost,move,r-1,c) + dCost;
      int mDist = editDistance(s1,s2,cost,move,r-1,c-1) +
                  (s1[r-1] == s2[c-1] ? 0 : mCost);
      if ( iDist < dDist ) {
        if ( iDist < mDist ) { // insertion is optima
          move.data[r][c] = 0;
          cost.data[r][c] = iDist;
        }
        else {
          move.data[r][c] = 2; // match is optimal
          cost.data[r][c] = mDist;
        }
      }
```

editDistance() algorithm

editDistance.cpp

```
      else {
        if ( dDist < mDist ) {
          move.data[r][c] = 1; // deletion is optimal
          cost.data[r][c] = dDist;
        }
        else {
          move.data[r][c] = 2; // match is optimal
          cost.data[r][c] = mDist;
        }
      }
    }
  }
  return cost.data[r][c];
}
```


editDistance.cpp: printEdits()

editDistance.cpp

```
int printEdits(std::string& s1, std::string& s2, Matrix615<int>& move) {
  std::string o1, o2, m;    // output string and alignments
  int r = move.rowNums()-1;
  int c = move.colNums()-1;
  while( r >= 0 && c >= 0 && move.data[r][c] >= 0) { // back from the last character
    if ( move.data[r][c] == 0 ) {  // insertion
      o1 = "-" + o1;  o2 = s2[c-1] + o2;  m = "I" + m; --c;
    }
    else if ( move.data[r][c] == 1 ) { // delettion
      o1 = s1[r-1] + o1;  o2 = "-" + o2;  m = "D" + m; --r;
    }
    else if ( move.data[r][c] == 2 ) { // match or mismatch
      o1 = s1[r-1] + o1; o2 = s2[c-1] + o2;
      m = (s1[r-1] == s2[c-1] ? "-" : "*") + m;
      --r; --c;
    }
    else std::cout << r << " " << c << " " << move.data[r][c] << std::endl;
  }
  std::cout << m << std::endl << o1 << std::endl << o2 << std::endl;
}
```

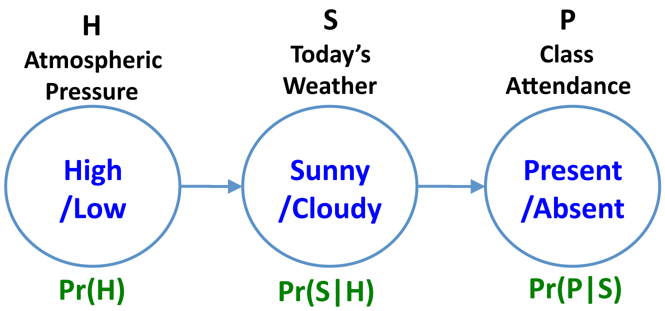

Running example

```
$ ./editDistance FOOD MONEY
EditDistance is 4
*-I**
FO-OD
MONEY
```


Graphical Model 101

- Graphical model is marriage between probability theory and graph theory (Michiael I. Jordan)
- Each random variable is represented as vertex
- Dependency between random variables is modeled as edge
  - Directed edge : conditional distribution
  - Undirected edge : joint distribution
- Unconnected pair of vertices (without path from one to another) is independent
- An effective tool to represent complex structure of dependence / independence between random variables.

An example graphical model



- Are H and P independent?
- Are H and P independent given S?

Example probability distribution

$\Pr(H)$

Value (H)	Description (H)	$\Pr(H)$
0	Low	0.3
1	High	0.7

$\Pr(S\|H)$

S	Description (S)	H	Description (H)	$\Pr(S\|H)$
0	Cloudy	0	Low	0.7
1	Sunny	0	Low	0.3
0	Cloudy	1	High	0.1
1	Sunny	1	High	0.9

Probability distribution (cont'd)

$\Pr(P\|S)$

P	Description (P)	S	Description (S)	$\Pr(P\|S)$
0	Absent	0	Cloudy	0.5
1	Present	0	Cloudy	0.5
0	Absent	1	Sunny	0.1
1	Present	1	Sunny	0.9

Full joint distribution

$\Pr(H, S, P)$

H	S	P	$\Pr(H, S, P)$
0	0	0	0.105
0	0	1	0.105
0	1	0	0.009
0	1	1	0.081
1	0	0	0.035
1	0	1	0.035
1	1	0	0.063
1	1	1	0.567

- With a full join distribution, any type of inference is possible
- As the number of variables grows, the size of full distribution table increases exponentially

$\Pr(H, P|S) = \Pr(H|S)\Pr(P|S)$

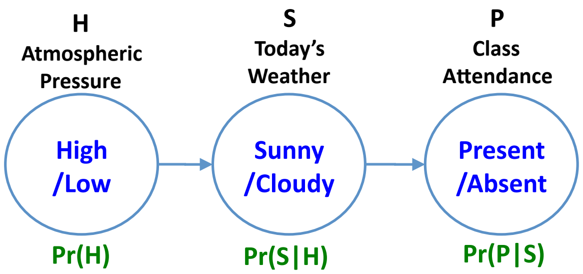
$\Pr(H, P|S)$

H	P	S	$\Pr(H, P\|S)$
0	0	0	0.3750
0	1	0	0.3750
1	0	0	0.1250
1	1	0	0.1250
0	0	1	0.0125
0	1	1	0.1125
1	0	1	0.0875
1	1	1	0.7875

$\Pr(H|S)$, $\Pr(P|S)$

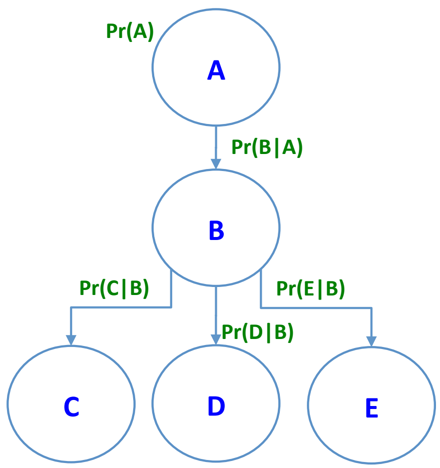
H	S	$\Pr(H\|S)$	P	S	$\Pr(P\|S)$
0	0	0.750	0	0	0.500
1	0	0.250	1	0	0.500
0	1	0.125	0	1	0.100
1	1	0.875	1	1	0.900

H and P are conditionally independent given S



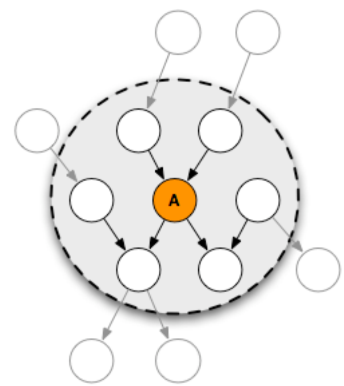
- H and P do not have direct path one from another
- All path from H to P is connected thru S.
- Conditioning on S separates H and P

Conditional independence in graphical models



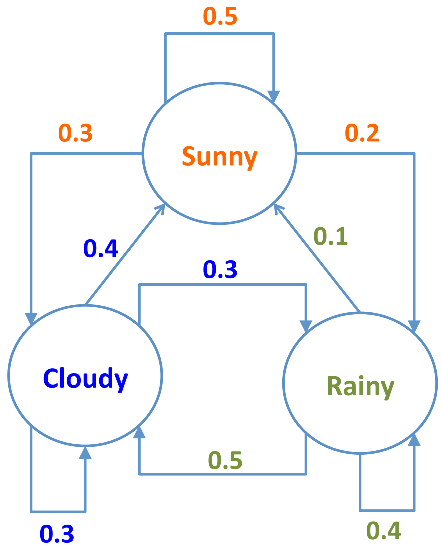
- $\Pr(A, C, D, E|B) = \Pr(A|B)\Pr(C|B)\Pr(D|B)\Pr(E|B)$

Markov Blanket



- If conditioned on the variables in the gray area (variables with direct dependency), A is independent of all the other nodes.
- $A \perp (U - A - \pi_A) | \pi_A$

Markov Process : An example



Mathematical representation of a Markov Process

$$
\begin{aligned}
\pi &= \begin{pmatrix} \Pr(q_1 = S_1 = \text{Sunny}) \\ \Pr(q_1 = S_2 = \text{Cloudy}) \\ \Pr(q_1 = S_3 = \text{Rainy}) \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.2 \\ 0.1 \end{pmatrix} \\
A_{ij} &= \Pr(q_{t+1} = S_j | q_t = S_i) \\
A &= \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.4 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}
\end{aligned}
$$

Example questions in Markov Process

What is the chance of rain in the day 2?

$$\Pr(q_2 = S_3) = (A^T \pi)_3 = 0.24$$

If it rains today, what is the chance of rain on the day after tomorrow?

$$\Pr(q_3 = S_3 | q_1 = S_3) = \left[(A^T)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]_3 = 0.33$$

Stationary distribution

$$
\begin{aligned}
\mathbf{p} &= A^T \mathbf{p} \\
p &= (0.346, 0.359, 0.295)^T
\end{aligned}
$$

Markov process is only dependent on the previous state

If it rains today, what is the chance of rain on the day after tomorrow?

$$\Pr(q_3 = S_3 | q_1 = S_3) = \left[(A^T)^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]_3 = 0.33$$

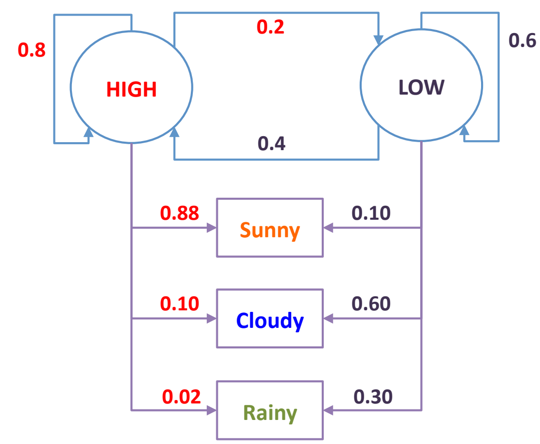
If it has rained for the past three days, what is the chance of rain on the day after tomorrow?

$$\Pr(q_5 = S_3 | q_1 = q_2 = q_3 = S_3) = \Pr(q_5 = S_3 | q_3 = S_3) = 0.33$$

Hidden Markov Models (HMMs)

- A Markov model where actual state is unobserved
  - Transition between states are probablistically modeled just like the Markov process
- Typically there are observable outputs associated with hidden states
  - The probability distribution of observable outputs given an hidden states can be obtained.

An example of HMM



- Direct Observation : (SUNNY, CLOUDY, RAINY)
- Hidden States : (HIGH, LOW)

Mathematical representation of the HMM example

States $S = \{S_1, S_2\} =$ (HIGH, LOW)

Outcomes $O = \{O_1, O_2, O_3\} =$ (SUNNY, CLOUDY, RAINY)

Initial States $\pi_i = \Pr(q_1 = S_i)$, $\pi = \{0.7, 0.3\}$

Transition $A_{ij} = \Pr(q_{t+1} = S_j | q_t = S_i)$

$$A = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}$$

Emission $B_{ij} = b_{q_t}(o_t) = b_{S_i}(O_j) = \Pr(o_t = O_j | q_t = S_i)$

$$B = \begin{pmatrix} 0.88 & 0.10 & 0.02 \\ 0.10 & 0.60 & 0.30 \end{pmatrix}$$

Unconditional marginal probabilities

What is the chance of rain in the day 4?

$$\mathbf{f}(\mathbf{q}_4) = \begin{pmatrix} \Pr(q_4 = S_1) \\ \Pr(q_4 = S_2) \end{pmatrix} = (A^T)^3 \pi = \begin{pmatrix} 0.669 \\ 0.331 \end{pmatrix}$$

$$\mathbf{g}(o_4) = \begin{pmatrix} \Pr(o_4 = O_1) \\ \Pr(o_4 = O_2) \\ \Pr(o_4 = O_3) \end{pmatrix} = B^T \mathbf{f}(\mathbf{q}_4) = \begin{pmatrix} 0.621 \\ 0.266 \\ 0.233 \end{pmatrix}$$

The chance of rain in day 4 is 23.3%

Summary

Edit Distance

- Alignment between two strings
- Can be converted to a problem similar to MTP

Hidden Markov Models

- Graphical models
- Conditional independence and Markov blankets
- Markov process
- Introduction to hidden Markov models

Next lectures

- More hidden Markov Models