

Biostatistics 602 - Statistical Inference

Lecture 18

Hypothesis Testing

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Last Lecture

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- What is the Asymptotic Relative Efficiency?
- What does mean that a statistic is asymptotically efficient?
- Is an MLE asymptotically efficient?

Asymptotic Normality

Definition: Asymptotic Normality

A statistic (or an estimator) $W_n(\mathbf{X})$ is *asymptotically normal* if

$$\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{d} \mathcal{N}(0, \nu(\theta))$$

for all θ

where \xrightarrow{d} stands for "converge in distribution"

- $\tau(\theta)$: "asymptotic mean"
- $\nu(\theta)$: "asymptotic variance"

We denote $W_n \sim \mathcal{AN}\left(\tau(\theta), \frac{\nu(\theta)}{n}\right)$.

Central Limit Theorem

Central Limit Theorem

Assume $X_i \stackrel{\text{i.i.d.}}{\sim} f(x|\theta)$ with finite mean $\mu(\theta)$ and variance $\sigma^2(\theta)$.

$$\bar{X} \sim \mathcal{N}\left(\mu(\theta), \frac{\sigma^2(\theta)}{n}\right)$$
$$\Leftrightarrow \sqrt{n}(\bar{X} - \mu(\theta)) \xrightarrow{d} \mathcal{N}(0, \sigma^2(\theta))$$

Theorem 5.5.17 - Slutsky's Theorem

If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$, where a is a constant,

- 1 $Y_n \cdot X_n \xrightarrow{d} aX$
- 2 $X_n + Y_n \xrightarrow{d} X + a$

Delta Method

Theorem 5.5.24 - Delta Method

Assume $W_n \sim \mathcal{AN}\left(\theta, \frac{\nu(\theta)}{n}\right)$. If a function g satisfies $g'(\theta) \neq 0$, then

$$g(W_n) \sim \mathcal{AN}\left(g(\theta), [g'(\theta)]^2 \frac{\nu(\theta)}{n}\right)$$

Asymptotic Efficiency

Definition : Asymptotic Efficiency for iid samples

A sequence of estimators W_n is asymptotically efficient for $\tau(\theta)$ if for all $\theta \in \Omega$,

$$\sqrt{n}(W_n - \tau(\theta)) \xrightarrow{d} \mathcal{N}\left(0, \frac{[\tau'(\theta)]^2}{I(\theta)}\right)$$

$$\iff W_n \sim \mathcal{AN}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{nI(\theta)}\right)$$

$$I(\theta) = E\left[\left\{\frac{\partial}{\partial\theta} \log f(X|\theta)\right\}^2 \mid \theta\right]$$

$$= -E\left[\frac{\partial^2}{\partial\theta^2} \log f(X|\theta) \mid \theta\right] \quad (\text{if interchangeability holds})$$

Note: $\frac{[\tau'(\theta)]^2}{nI(\theta)}$ is the C-R bound for unbiased estimators of $\tau(\theta)$.

Asymptotic Efficiency of MLEs

Theorem 10.1.12

Let X_1, \dots, X_n be iid samples from $f(x|\theta)$. Let $\hat{\theta}$ denote the MLE of θ . Under same regularity conditions, $\hat{\theta}$ is consistent and asymptotically normal for θ , i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I(\theta)}\right) \text{ for every } \theta \in \Omega$$

And if $\tau(\theta)$ is continuous and differentiable in θ , then

$$\begin{aligned}\sqrt{n}(\hat{\theta} - \theta) &\xrightarrow{d} \mathcal{N}\left(0, \frac{[\tau'(\theta)]}{I(\theta)}\right) \\ \implies \tau(\hat{\theta}) &\sim \mathcal{AN}\left(\tau(\theta), \frac{[\tau'(\theta)]^2}{nI(\theta)}\right)\end{aligned}$$

Again, note that the asymptotic variance of $\tau(\hat{\theta})$ is Cramer-Rao lower bound for unbiased estimators of $\tau(\theta)$.

Hypothesis Testing

Definition

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Two complementary statements about θ

- Null hypothesis : $H_0 : \theta \in \Omega_0$
- Alternative hypothesis : $H_1 : \theta \in \Omega_0^c$

$$\theta \in \Omega = \Omega \cup \Omega^c.$$

Simple and composite hypothesis

Simple hypothesis

Both H_0 and H_1 consist of only one parameter value.

- $H_0 : \theta = \theta_0 \in \Omega_0$
- $H_1 : \theta = \theta_1 \in \Omega_0^c$

Another Example of Hypothesis

- Let θ denotes the proportion of defective items from a machine.
- One may want the proportion to be less than a specified maximum acceptable proportion θ_0 .
- We want to test whether the products produced by the machine is acceptable.

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- We want to test whether the products produced by the machine is acceptable.

$$H_0 : \theta \leq \theta_0 \quad (\text{acceptable})$$

$$H_1 : \theta > \theta_0 \quad (\text{unacceptable})$$

Hypothesis Testing Procedure

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- ① For which sample points H_0 is accepted as true (the subset of the sample space for which H_0 is accepted is called the acceptable region).
- ② For which sample points H_0 is rejected and H_1 is accepted as true (the subset of sample space for which H_0 is rejected is called the rejection region or critical region).

Rejection region (R) on a hypothesis is usually defined through a test statistic $W(\mathbf{X})$. For example,

$$R_1 = \{\mathbf{x} : W(\mathbf{x}) > c, \mathbf{x} \in \mathcal{X}\}$$

$$R_2 = \{\mathbf{x} : W(\mathbf{x}) \leq c, \mathbf{x} \in \mathcal{X}\}$$

Example of hypothesis testing

$X_1, X_2, X_3 \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$. The hypothesis test

$$H_0 : p \leq 0.5$$

$$H_1 : p > 0.5$$

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- Test 1 : Reject H_0 if $\mathbf{x} \in \{(1, 1, 1)\}$
 \iff rejection region = $\{(1, 1, 1)\}$
 \iff rejection region = $\{\mathbf{x} : \sum x_i > 2\}$

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- Test 2 : Reject H_0 if $\mathbf{x} \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$
 \iff rejection region = $\{(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$
 \iff rejection region = $\{\mathbf{x} : \sum x_i > 1\}$

Example

Let X_1, \dots, X_n be changes in blood pressure after a treatment.

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		Decision	
		Accept H_0	Reject H_0
Truth	H_0	Correct Decision	Type I error
	H_1	Type II error	Correct Decision

Type I and Type II error

Type I error

If $\theta \in \Omega_0$ (if the null hypothesis is true), the probability of making a type I error is

$$\Pr(\bar{X} \in R | \theta)$$

Type I and Type II error

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If $\theta \in \Omega_0$ (if the null hypothesis is true), the probability of making a type I error is

$$\Pr(\bar{X} \in R|\theta)$$

Type II error

If $\theta \in \Omega_0^c$ (if the alternative hypothesis is true), the probability of making a type II error is

$$\Pr(\bar{X} \notin R|\theta) = 1 - \Pr(\bar{X} \in R|\theta)$$

Power function

Definition - The power function

The power function of a hypothesis test with rejection region R is the function of θ defined by

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- Probability of type I error = $\beta(\theta)$ if $\theta \in \Omega_0$.
- Probability of type II error = $1 - \beta(\theta)$ if $\theta \in \Omega_0^c$.

An ideal test should have power function satisfying $\beta(\theta) = 0$ for all $\theta \in \Omega_0$, $\beta(\theta) = 1$ for all $\theta \in \Omega_0^c$, which is typically not possible in practice.

Example of power function

Problem

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Test 1 rejects H_0 if and only if all "success" are observed. i.e.

$$R = \{\mathbf{x} : \mathbf{x} = (1, 1, 1, 1, 1)\}$$

$$= \left\{ \mathbf{x} : \sum_{i=1}^5 x_i = 5 \right\}$$

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- 1 Compute the power function
- 2 What is the maximum probability of making type I error?
- 3 What is the probability of making type II error if $\theta = 2/3$?

Solution for Test 1

Power function

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$$\max_{\theta \in (0, 0.5]} \beta(\theta) = \max_{\theta \in (0, 0.5]} \theta^5 = 0.5^5 = 1/32 \approx 0.031$$

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Type II error when $\theta = 2/3$

$$1 - \beta(\theta) |_{\theta = \frac{2}{3}} = 1 - \theta^5 |_{\theta = \frac{2}{3}} = 1 - (2/3)^5 = 57/81 \approx 0.704$$

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Test 2 rejects H_0 if and only if 3 or more "success" are observed. i.e.

$$R = \left\{ \mathbf{x} : \sum_{i=1}^5 x_i \geq 3 \right\}$$

Another Example

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- 1 Compute the power function
- 2 What is the maximum probability of making type I error?
- 3 What is the probability of making type II error if $\theta = 2/3$?

Solution for Test 2

Power function

$$\beta(\theta) = \Pr(\sum X_i \geq 3|\theta) = \binom{5}{3}\theta^3(1-\theta)^2 + \binom{5}{4}\theta^4(1-\theta) + \binom{5}{5}\theta^5$$

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Power function

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Maximum type I error

We need to find the maximum of $\beta(\theta)$ for $\theta \in \Omega_0 = (0, 0.5]$

$$\beta'(\theta) = 30\theta^2(\theta - 1)^2 > 0$$

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Maximum type I error

We need to find the maximum of $\beta(\theta)$ for $\theta \in \Omega_0 = (0, 0.5]$

$$\beta'(\theta) = 30\theta^2(\theta - 1)^2 > 0$$

$\beta(\theta)$ is increasing in $\theta \in (0, 1)$. Maximum type I error is $\beta(0.5) = 0.5$

Type II error when $\theta = 2/3$

$$1 - \beta(\theta) |_{\theta = \frac{2}{3}} = 1 - \theta^5 |_{\theta = \frac{2}{3}} = 1 - (2/3)^5 \approx 0.21$$

Sizes and Levels of Tests

Size α test

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Level α test

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Level α test

A test with power function $\beta(\theta)$ is a level α test if

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Any size α test is also a level α test

Revisiting Previous Examples

Test 1

$$\sup_{\theta \in \Omega_0} \beta(\theta) = \sup_{\theta \in \Omega_0} \theta^5 = 0.5^5 = 0.03125$$

The size is 0.03125, and this is a level 0.05 test, or a level 0.1 test, but not a level 0.01 test.

Revisiting Previous Examples

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The size is 0.03125, and this is a level 0.05 test, or a level 0.1 test, but not a level 0.01 test.

Test 2

$$\sup_{\theta \in \Omega_0} \beta(\theta) = 0.5$$

The size is 0.5

Constructing a good test

- 1 Construct all the level α test.

Constructing a good test

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- 2 Within this level of tests, we search for the test with Type II error probability as small as possible; equivalently, we want the test with the largest power if $\theta \in \Omega_0^c$.

Review on standard normal and t distribution

Quantile of standard normal distribution

Let $Z \sim \mathcal{N}(0, 1)$ with pdf $f_Z(z)$ and cdf $F_Z(z)$. The α -th quantile z_α or $(1 - \alpha)$ -th quantile $z_{1-\alpha}$ of the standard distribution satisfy

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$$z_{1-\alpha} = -z_\alpha$$

Quantile of t distribution

Let $T \sim t_{n-1}$ with pdf $f_{T,n-1}(t)$ and cdf $F_{T,n-1}(t)$. The α -th quantile $t_{n-1,\alpha}$ or $(1 - \alpha)$ -th quantile $t_{n-1,1-\alpha}$ of the standard distribution satisfy

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Quantile of t distribution

Let $T \sim t_{n-1}$ with pdf $f_{T,n-1}(t)$ and cdf $F_{T,n-1}(t)$. The α -th quantile $t_{n-1,\alpha}$ or $(1 - \alpha)$ -th quantile $t_{n-1,1-\alpha}$ of the standard distribution satisfy

$$\Pr(T \geq t_{n-1,\alpha}) = \alpha \quad \text{or} \quad t_{n-1,\alpha} = F_{T,n-1}^{-1}(1 - \alpha)$$

$$\Pr(T \leq t_{n-1,1-\alpha}) = \alpha \quad \text{or} \quad t_{n-1,1-\alpha} = F_{T,n-1}^{-1}(\alpha)$$

$$t_{n-1,1-\alpha} = -t_{n-1,\alpha}$$

Likelihood Ratio Tests (LRT)

Definition

Let $L(\theta|\mathbf{x})$ be the likelihood function of θ . The likelihood ratio test statistic for testing $H_0 : \theta \in \Omega_0$ vs. $H_1 : \theta \in \Omega_0^c$ is

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$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Omega_0} L(\theta|\mathbf{x})}{\sup_{\theta \in \Omega} L(\theta|\mathbf{x})} = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})}$$

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The *likelihood ratio test* is a test that rejects H_0 if and only if $\lambda(\mathbf{x}) \leq c$ where $0 \leq c \leq 1$.

Properties of LRT

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 - If $c = 1$, null hypothesis will always be rejected.
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Then we get a size α test.

Example of LRT

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For the LRT test and its power function

Solution

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We need to find MLE of θ over $\Omega = (-\infty, \infty)$ and $\Omega_0 = (-\infty, \theta_0]$.

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To maximize $L(\theta|\mathbf{x})$, we need to maximize $\exp\left[-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2\sigma^2}\right]$, or equivalently to minimize $\sum_{i=1}^n (x_i - \theta)^2$.

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The equation above minimizes when $\theta = \hat{\theta} = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$.

MLE of θ over $\Omega_0 = (-\infty, \theta_0)$

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To summarize,

$$\hat{\theta}_0 = \begin{cases} \bar{X} & \text{if } \bar{X} \leq \theta_0 \\ \theta_0 & \text{if } \bar{X} > \theta_0 \end{cases}$$

Likelihood ratio test

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_0|\mathbf{x})}{L(\hat{\theta}|\mathbf{x})} = \begin{cases} 1 & \text{if } \bar{X} \leq \theta_0 \\ \frac{\exp\left[-\frac{\sum_{i=1}^n (x_i - \theta_0)^2}{2\sigma^2}\right]}{\exp\left[-\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{2\sigma^2}\right]} & \text{if } \bar{X} > \theta_0 \end{cases}$$

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Therefore, the likelihood test rejects the null hypothesis if and only if

$$\exp\left[-\frac{n(\bar{x} - \theta_0)^2}{2\sigma^2}\right] \leq c$$

and $\bar{x} \geq \theta_0$.

Specifying c

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Specifying c (cont'd)

So, LRT rejects H_0 if and only if

$$\begin{aligned} \bar{x} - \theta_0 &\geq \sqrt{\frac{2\sigma^2 \log c}{n}} \\ \Leftrightarrow \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} &\geq \frac{\sqrt{\frac{2\sigma^2 \log c}{n}}}{\sigma/\sqrt{n}} = c^* \end{aligned}$$

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Therefore, the rejection region is

$$\left\{ \bar{x} : \frac{\bar{x} - \theta_0}{\sigma/\sqrt{n}} \geq c^* \right\}$$

