Biostatistics 602 - Statistical Inference
Lecture 06
Basu's Theorem

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Last Lecture

1. What is a complete statistic?

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2. Why it is called as "complete statistic"?
3. Can the same statistic be both complete and incomplete statistics, depending on the parameter space?
4. What is the relationship between complete and sufficient statistics?
5. Is a minimal sufficient statistic always complete?

Complete Statistics

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- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
  - In other words, $g(T) = 0$ almost surely.
- Equivalently, $T(\mathbf{X})$ is called a *complete statistic*

Example - Poisson distribution

When parameter space is limited - NOT complete

- Suppose $\mathcal{T} = \left\{ f_T : f_T(t|\lambda) = \frac{\lambda^t e^{-\lambda}}{t!} \right\}$ for $t \in \{0, 1, 2, \cdots\}$. Let $\lambda \in \Omega = \{1, 2\}$. This family is NOT complete

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With full parameter space - complete

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda), \lambda > 0.$
- $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete statistic.

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$.

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Solution

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta \in \mathbb{N}$.
Note that $f_X(x) = \frac{1}{\theta} I(x \in \{1, \cdots, \theta\}) = \frac{1}{\theta} I_{\mathbb{N}_\theta}(x)$.

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$$E[g(T)|\theta] = E[g(X)|\theta] = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0$$

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for all $\theta \in \mathbb{N}$, which implies

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- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$
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- $\vdots$
- if $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + \cdots + g(k-1) + g(2) = g(k) = 0$.

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- $\vdots$
- if $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + \cdots + g(k-1) + g(2) = g(k) = 0$.

Therefore, $g(x) = 0$ for all $x \in \mathbb{N}$, and $T(X) = X$ is a complete statistic for $\theta \in \Omega = \mathbb{N}$.

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$.

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Solution

Define a nonzero $g(x)$ as follows

$$g(x) = \begin{cases} 1 & x = n \\ -1 & x = n+1 \\ 0 & \text{otherwise} \end{cases}$$

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$$E[g(T)|\theta] = \frac{1}{\theta}\sum_{x=1}^{\theta} g(x) = \begin{cases} 0 & \theta \neq n \\ \frac{1}{\theta} & \theta = n \end{cases}$$

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Because Ω does not include n, $g(x) = 0$ for all $\theta \in \Omega = \mathbb{N} - \{n\}$, and $T(X) = X$ is not a complete statistic.

Last Lecture : Ancillary and Complete Statistics

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.
- Is $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ a complete statistic?

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A Simple Proof

- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.

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- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.
- Define $g(\mathbf{T}) = X_{(n)} - X_{(1)} - E(R)$. Note that $E(R)$ is constant to θ.

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- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.
- Define $g(\mathbf{T}) = X_{(n)} - X_{(1)} - E(R)$. Note that $E(R)$ is constant to θ.
- Then $E[g(\mathbf{T})|\theta] = E(R) - E(R) = 0$, so T is not a complete statistic.

Useful Fact 1 : Ancillary and Complete Statistics

Fact

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Proof

Define $g(T) = r(T) - E[r(T)]$, which does not depend on the parameter θ because $r(T)$ is ancillary. Then $E[g(T)|\theta] = 0$ for a non-zero function $g(T)$, and $T(\mathbf{X})$ is not a complete statistic.

Useful Fact 2 : Arbitrary Function of Complete Statistics

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Assume that $E[g(T^*)|\theta] = 0$ for all θ, then $E[g \circ r(T)|\theta] = 0$ holds for all θ too. Because $T(\mathbf{X})$ is a complete statistic, $\Pr[g \circ r(T) = 0] = 1, \ \forall \theta \in \Omega$. Therefore $\Pr[g(T^*) = 0] = 1$, and T^* is a complete statistic.

Theorem 6.2.28 - Lehman and Schefle (1950)

The textbook version

If a minimal sufficient statistic exists, then any complete statistic is also a minimal sufficient statistic.

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The converse is NOT true

A minimal sufficient statistic is not necessarily complete. (Recall the example in the last lecture).

Basu's Theorem

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Proof strategy - for discrete case

Suppose that $S(\mathbf{X})$ is an ancillary statistic. We want to show that

$$\Pr(S(\mathbf{X}) = s | T(\mathbf{X}) = t) = \Pr(S(\mathbf{X}) = s), \ \forall t \in \mathcal{T}$$

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$$\begin{aligned} \Pr(T(\mathbf{X}) = t | S(\mathbf{X}) = s) &= \Pr(T(\mathbf{X}) = t) \\ \Pr(T(\mathbf{X}) = t \wedge S(\mathbf{X}) = s) &= \Pr(T(\mathbf{X}) = t) \Pr(S(\mathbf{X}) = s) \end{aligned}$$

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- As $S(\mathbf{X})$ is ancillary, by definition, it does not depend on θ.
- As $T(\mathbf{X})$ is sufficient, by definition, $f_{\mathbf{X}}(\mathbf{X}|T(\mathbf{X}))$ is independent of θ.
- Because $S(\mathbf{X})$ is a function of $\mathbf{X}$, $\Pr(S_{\mathbf{X}}|T(\mathbf{X}))$ is also independent of θ.

Proof of Basu's Theorem

- As $S(\mathbf{X})$ is ancillary, by definition, it does not depend on θ.
- As $T(\mathbf{X})$ is sufficient, by definition, $f_{\mathbf{X}}(\mathbf{X}|T(\mathbf{X}))$ is independent of θ.
- Because $S(\mathbf{X})$ is a function of $\mathbf{X}$, $\Pr(S_{\mathbf{X}}|T(\mathbf{X}))$ is also independent of θ.
- We need to show that $\Pr(S(\mathbf{X}) = s|T(\mathbf{X}) = t) = \Pr(S(\mathbf{X}) = s),\ \forall t \in \mathcal{T}$.

Proof of Basu's Theorem (cont'd)

$$\Pr(S(\mathbf{X}) = s|\theta) = \sum_{t \in \mathcal{T}} \Pr(S(\mathbf{X}) = s|\, T(\mathbf{X}) = t) \Pr(\, T(\mathbf{X}) = t|\theta) \quad (1)$$

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Define $g(t) = \Pr(S(\mathbf{X}) = s|\, T(\mathbf{X}) = t) - \Pr(S(\mathbf{X}) = s)$.

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Define $g(t) = \Pr(S(\mathbf{X}) = s|\, T(\mathbf{X}) = t) - \Pr(S(\mathbf{X}) = s)$. Taking (1)-(3),

$$\sum_{t \in \mathcal{T}} [\Pr(S(\mathbf{X}) = s|\, T(\mathbf{X}) = t) - \Pr(S(\mathbf{X}) = s)] \Pr(\, T(\mathbf{X}) = t|\theta) = 0$$

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$$\Pr(S(\mathbf{X}) = s|\theta) = \sum_{t\in\mathcal{T}} \Pr(S(\mathbf{X}) = s|\, T(\mathbf{X}) = t)\Pr(\, T(\mathbf{X}) = t|\theta) \quad (1)$$

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$$\sum_{t\in\mathcal{T}} g(t)\Pr(\, T(\mathbf{X}) = t|\theta) = E[g(\, T(\mathbf{X}))|\theta] = 0$$

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Because $T(\mathbf{X})$ is a complete statistic,

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$$\sum_{t\in\mathcal{T}} g(t)\Pr(\, T(\mathbf{X}) = t|\theta) = E[g(\, T(\mathbf{X}))|\theta] = 0$$

Because $T(\mathbf{X})$ is a complete statistic, it implies that $g(t) = 0$ almost surely for all possible $t \in \mathcal{T}$

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Because $T(\mathbf{X})$ is a complete statistic, it implies that $g(t) = 0$ almost surely for all possible $t \in \mathcal{T}$. Therefore, $S(\mathbf{X})$ is independent of $T(\mathbf{X})$.

Application of Basu's Theorem

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- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$.
- Calculate $E\left[\frac{X_{(1)}}{X_{(n)}}\right]$ and $E\left[\frac{X_{(1)}+X_{(2)}}{X_{(n)}}\right]$

Application of Basu's Theorem

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- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta)$.
- Calculate $E\left[\frac{X_{(1)}}{X_{(n)}}\right]$ and $E\left[\frac{X_{(1)}+X_{(2)}}{X_{(n)}}\right]$

A strategy for the solution

- We know that $X_{(n)}$ is sufficient statistic.

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- We know that $X_{(n)}$ is complete, too.
- We can easily show that $X_{(1)}/X_{(n)}$ is an ancillary statistic.
- Then we can leverage Basu's Theorem for the calculation.

Showing that $X_{(1)}/X_{(n)}$ is Ancillary

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Because the distribution of $Y_1, \cdots, Y_n$ does not depend on θ, $X_{(1)}/X_{(n)}$ is an ancillary statistic for θ.

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Exponential Family

Definition 3.4.1

The random variable X belongs to an exponential family of distributions, if its pdf/pmf can be written in the form

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Show that a Poisson(λ) ($\lambda > 0$) belongs to the exponential family

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$$f_X(x|\boldsymbol{\theta} = (\mu, \sigma^2)) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

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A Specialized Normal Distribution : $\mathcal{N}(\mu, \mu^2)$

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\begin{aligned}
f_X(x|\mu) &= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left[-\frac{(x-\mu)^2}{2\mu^2}\right] \\
&= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left[-\frac{x^2}{2\mu^2} + \frac{2\mu x}{2\mu^2} - \frac{\mu^2}{2\mu^2}\right] \\
&= \frac{1}{\sqrt{2\pi\mu^2}} \exp\left(-\frac{1}{2}\right) \exp\left[-\frac{1}{2\mu^2}x^2 + \frac{1}{\mu}x\right]
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$$
f_X(x|\mu) = h(x)c(\mu)\exp\left[\sum_{j=1}^{k} w_j(\mu)t_j(x)\right]
$$

Summary

Today

- More on complete statistics
- Basu's Theorem
- Exponential Family

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Next Lecture

- More on Exponential Family