# Biostatistics 602 - Statistical Inference
# Lecture 16
# Evaluation of Bayes Estimator

Hyun Min Kang

March 14th, 2013

# Last Lecture

- What is a Bayes Estimator?

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- Compared to other estimators, what are advantages of Bayes Estimator?
- What is conjugate family?
- What are the conjugate families of Binomial, Poisson, and Normal distribution?

# Recap - Bayes Estimator

- $\theta$ : parameter
- $\pi(\theta)$ : prior distribution
- $\mathbf{X}|\theta \sim f_{\mathbf{X}}(\mathbf{x}|\theta)$ : sampling distribution
- Posterior distribution of $\theta|\mathbf{x}$

$$\pi(\theta|\mathbf{x}) = \frac{\text{Joint}}{\text{Marginal}} = \frac{f_{\mathbf{X}}(\mathbf{x}|\theta)\pi(\theta)}{m(\mathbf{x})}$$

$$m(\mathbf{x}) = \int f(\mathbf{x}|\theta)\pi(\theta)\,d\theta \qquad \text{(Bayes' rule)}$$

- Bayes Estimator of $\theta$ is

$$\mathrm{E}(\theta|\mathbf{x}) = \int_{\theta\in\Omega} \theta\pi(\theta|\mathbf{x})\,d\theta$$

# Recap - Example

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$
- $\pi(p) \sim \text{Beta}(\alpha, \beta)$
- Prior guess : $\hat{p} = \frac{\alpha}{\alpha+\beta}$.
- Posterior distribution : $\pi(p|\mathbf{x}) \sim \text{Beta}(\sum x_i + \alpha, n - \sum x_i + \beta)$
- Bayes estimator

$$\hat{p} = \frac{\alpha + \sum x_i}{\alpha + \beta + n} = \frac{\sum x_i}{n} \frac{n}{\alpha + \beta + n} + \frac{\alpha}{\alpha + \beta} \frac{\alpha + \beta}{\alpha + \beta + n}$$

# Loss Function Optimality

The mean squared error (MSE) is defined as

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Let $\hat{\theta}$ is an estimator.

- If $\hat{\theta} = \theta$, it makes a correct decision and loss is 0
- If $\hat{\theta} \neq \theta$, it makes a mistake and loss is not 0.

# Loss Function

Let $L(\theta, \hat{\theta})$ be a function of $\theta$ and $\hat{\theta}$.

- Squared error loss

$$
\begin{aligned}
L(\hat{\theta}, \theta) &= (\hat{\theta} - \theta)^2 \\
\text{MSE} &= \text{Average Loss} = \mathrm{E}[L(\theta, \hat{\theta})]
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- Absolute error loss

$$L(\hat{\theta}) = |\hat{\theta} - \theta|$$

- A loss that penalties overestimation more than underestimation

$$L(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2 I(\hat{\theta} < \theta) + 10(\hat{\theta} - \theta)^2 I(\hat{\theta} \geq \theta)$$

# Risk Function - Average Loss

$$R(\theta, \hat{\theta}) = \mathrm{E}[L(\theta, \hat{\theta}(\mathbf{X}))|\theta]$$

If $L(\theta, \hat{\theta}) = (\hat{\theta} - \theta)^2$, $R(\theta, \hat{\theta})$ is MSE.

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Bayes risk is defined as the average risk across all values of $\theta$ given prior $\pi(\theta)$

$$\int_\Omega R(\theta, \hat{\theta})\pi(\theta)\, d\theta$$

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The Bayes rule with respect to a prior $\pi$ is the optimal estimator with respect to a Bayes risk, which is defined as the one that minimize the Bayes risk.

# Alternative definition of Bayes Risk

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&= \int_\Omega \left[\int_\mathcal{X} \pi(\theta|\mathbf{x})m(\mathbf{x})L(\theta, \hat{\theta}(\mathbf{x}))\,d\mathbf{x}\right]d\theta \\
&= \int_\mathcal{X} \left[\int_\Omega L(\theta, \hat{\theta}(X))\pi(\theta|\mathbf{x})\,d\theta\right]m(\mathbf{x})\,d\mathbf{x}
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# Posterior Expected Loss

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An alternative definition of Bayes rule estimator is the estimator that minimizes the posterior expected loss.

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So, the goal is to minimize $\mathrm{E}[(\theta - \hat{\theta})^2 | \mathbf{X} = \mathbf{x}]$

$$\mathrm{E}\left[(\theta - \hat{\theta})^2 | \mathbf{X} = \mathbf{x}\right] = \mathrm{E}\left[(\theta - \mathrm{E}(\theta|\mathbf{x}) + \mathrm{E}(\theta|\mathbf{x}) - \hat{\theta})^2 | \mathbf{X} = \mathbf{x}\right]$$

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which is minimized when $\hat{\theta} = \mathrm{E}(\theta|\mathbf{x})$.

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Loss function $L(\theta, \hat{\theta})$

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**Bayes risk** Average risk across all $\theta$, based on the prior of $\theta$.

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**Bayes estimator** $\hat{\theta} = \mathrm{E}[\theta|\mathbf{x}]$. Based on squared error loss,

- Minimize Bayes risk
- Minimize Posterior Expected Loss

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$\frac{\partial}{\partial \hat{\theta}} \mathrm{E}[L(\theta, \hat{\theta}(\mathbf{x}))] = 0$, and $\hat{\theta}$ is posterior median.

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For absolute error loss, the posterior expected loss is $\mathrm{E}(|\theta - \hat{\theta}||\mathbf{X} = \mathbf{x})$. As shown previously, this is minimized by choosing $\hat{\theta}$ as the median of $\pi(\theta|\mathbf{x})$.

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- Bayes estimator that minimizes posterior expected absolute error loss is the posterior median

$$\int_0^{\hat{\theta}} \frac{\Gamma(\alpha + \beta + n)}{\Gamma(\sum x_i + \alpha)\Gamma(n - \sum x_i + \beta)} p^{\sum x_i + \alpha - 1}(1-p)^{n - \sum x_i + \beta - 1} dp = \frac{1}{2}$$

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## Definition - Consistency

Let $W_n = W_n(X_1, \cdots, X_n) = W_n(\mathbf{X})$ be a sequence of estimators for $\tau(\theta)$. We say $W_n$ is consistent for estimating $\tau(\theta)$ if $W_n \xrightarrow{\text{P}} \tau(\theta)$ under $P_\theta$ for every $\theta \in \Omega$.

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$W_n \xrightarrow{\text{P}} \tau(\theta)$ (converges in probability to $\tau(\theta)$) means that, given any $\epsilon > 0$.

$$\begin{aligned} \lim_{n\rightarrow\infty} \Pr(|W_n - \tau(\theta)| \geq \epsilon) &= 0 \\ \lim_{n\rightarrow\infty} \Pr(|W_n - \tau(\theta)| < \epsilon) &= 0 \end{aligned}$$

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When $|W_n - \tau(\theta)| < \epsilon$ can also be represented that $W_n$ is close to $\tau(\theta)$. Consistency implies that the probability of $W_n$ close to $\tau(\theta)$ approaches to 1 as $n$ goes to $\infty$.

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$$\begin{aligned}\Pr(|W_n - \tau(\theta)| \geq \epsilon) &= \Pr((W_n - \tau(\theta))^2 \geq \epsilon^2) \\ &\leq \frac{\mathrm{E}[W_n - \tau(\theta)]^2}{\epsilon^2} \\ &= \frac{\mathrm{MSE}(W_n)}{\epsilon^2} = \frac{\mathrm{Bias}^2(W_n) + \mathrm{Var}(W_n)}{\epsilon^2}\end{aligned}$$

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&= \frac{\mathrm{MSE}(W_n)}{\epsilon^2} = \frac{\mathrm{Bias}^2(W_n) + \mathrm{Var}(W_n)}{\epsilon^2}
\end{aligned}
$$

Need to show that both $\mathrm{Bias}(W_n)$ and $\mathrm{Var}(W_n)$ converges to zero

# Theorem for consistency

## Theorem 10.1.3

If $W_n$ is a sequence of estimators of $\tau(\theta)$ satisfying

- $\lim_{n->\infty} \text{Bias}(W_n) = 0.$
- $\lim_{n->\infty} \text{Var}(W_n) = 0.$

for all $\theta$, then $W_n$ is consistent for $\tau(\theta)$

# Weak Law of Large Numbers

**Theorem 5.5.2**

Let $X_1, \cdots, X_n$ be iid random variables with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Then $\overline{X}_n$ converges in probability to $\mu$.
i.e. $\overline{X}_n \xrightarrow{\mathrm{P}} \mu$.

# Consistent sequence of estimators

## Theorem 10.1.5

Let $W_n$ is a consistent sequence of estimators of $\tau(\theta)$. Let $a_n$, $b_n$ be sequences of constants satisfying

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## Continuous Map Theorem

If $W_n$ is consistent for $\theta$ and $g$ is a continuous function, then $g(W_n)$ is consistent for $g(\theta)$.

# Example

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1. Show that $\overline{X}_n$ is consistent for $\mu$.
2. Show that $\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2$ is consistent for $\sigma^2$.
3. Show that $\frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$ is consistent for $\sigma^2$.

# Example - Solution

**Proof: $\overline{X}_n$ is consistent for $\mu$**

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- $\text{Var}(\overline{X}_n) = \text{Var}\left(\frac{\sum_{i=1}^n X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i) = \sigma^2/n.$

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- $\lim_{n\to\infty} \text{Var}(\overline{X}) = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$.

By Theorem 10.1.3. $\overline{X}$ is consistent for $\mu$.

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Note that $\overline{X}^2$ is a function of $\overline{X}$. Define $g(x) = x^2$, which is a continuous function. Then $\overline{X}^2 = g(\overline{X})$ is consistent for $\mu^2$.

# Solution - consistency for $\sigma^2$

$$\begin{aligned}\frac{\sum(X_i - \overline{X})^2}{n} &= \frac{\sum(X_i^2 + \overline{X}^2 - 2X_i\overline{X})}{n} \\ &= \frac{\sum X_i^2 + n\overline{X}^2 - 2\overline{X}\sum_{i=1}^n X_i}{n} \\ &= \frac{\sum X_i^2}{n} - \overline{X}^2\end{aligned}$$

By law of large numbers,

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Note that $\overline{X}^2$ is a function of $\overline{X}$. Define $g(x) = x^2$, which is a continuous function. Then $\overline{X}^2 = g(\overline{X})$ is consistent for $\mu^2$. Therefore,

$$\frac{\sum(X_i - \overline{X}_n)^2}{n} = \frac{\sum X_i^2}{n} - \overline{X}^2 \xrightarrow{\text{P}} (\mu^2 + \sigma^2) - \mu^2 = \sigma^2$$

# Solution - consistency for $\sigma^2$ (cont'd)

Define $S_n^2 = \frac{1}{n-1}\sum(X_i - \overline{X}_n)^2$, and $(S_n^*)^2 = \frac{1}{n}\sum(X_i - \overline{X}_n)^2$.

# Solution - consistency for $\sigma^2$ (cont'd)

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Because $(S_n^*)^2$ was shown to be consistent for $\sigma^2$ previously, and $a_n = \frac{n}{n-1} \rightarrow 1$ as $n \rightarrow \infty$, by Theorem 10.1.5, $S_n^2$ is also consistent for $\sigma^2$.

# Example - Exponential Family

## Problem

Suppose $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\beta)$.

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Suppose $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\beta)$.

1. Propose a consistent estimator of the median.
2. Propose a consistent estimator of $\Pr(X \leq c)$ where $c$ is constant.

# Consistent estimator for the median

First, we need to express the median in terms of the parameter $\beta$.

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By law of large numbers, $\overline{X}_n$ is consistent for $\mathrm{E}X = \beta$.
Applying continuous mapping Theorem to $g(x) = x \log 2$, $g(\overline{X}) = \overline{X}_n \log 2$ is consistent for $g(\beta) = \beta \log 2$ (median).

# Consistent estimator of $\Pr(X \leq c)$

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As $\overline{X}$ is consistent for $\beta$, $1 - e^{-c/\beta}$ is continuous function of $\beta$.

# Consistent estimator of $\Pr(X \leq c)$

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As $\overline{X}$ is consistent for $\beta$, $1 - e^{-c/\beta}$ is continuous function of $\beta$.
By continuous mapping Theorem, $g(\overline{X}) = 1 - e^{-c/\overline{X}}$ is consistent for $\Pr(X \leq c) = 1 - e^{-c/\beta} = g(\beta)$

# Consistent estimator of $\Pr(X \leq c)$ - Alternative Method

Define $Y_i = I(X_i \leq c)$. Then $Y_i \overset{\text{i.i.d.}}{\sim} \text{Bernoulli}(p)$ where $p = \Pr(X \leq c)$.

# Consistent estimator of $\Pr(X \leq c)$ - Alternative Method

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$$\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq c)$$

is consistent for $p$ by Law of Large Numbers.

# Summary

## Today

- Bayes Risk Functions
- Consistency
- Law of Large Numbers

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- Bayes Risk Functions
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## Next Lecture

- Central Limit Theorem
- Slutsky Theorem
- Delta Method