Biostatistics 602 - Statistical Inference
Lecture 22
p-Values

Hyun Min Kang

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Last Lecture

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- How about its asymptotic distribution? For testing which null/alternative hypotheses is the asymptotic distribution valid?
- What is a Wald Test?
- Describe a typical way to construct a Wald Test.

Asymptotics of LRT

Theorem 10.3.1

Consider testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Suppose $X_1, \cdots, X_n$ are iid samples from $f(x|\theta)$, and $\hat{\theta}$ is the MLE of θ, and $f(x|\theta)$ satisfies certain "regularity conditions" (e.g. see misc 10.6.2), then under H_0:

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$$-2 \log \lambda(\mathbf{x}) \xrightarrow{\text{d}} \chi^2_1$$

as $n \to \infty$.

Wald Test

Wald test relates point estimator of θ to hypothesis testing about θ.

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where θ_0 is the value of θ under H_0 and S_n is a consistent estimator of σ_W

p-Values

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Definition: p-Value

A *p-value* $p(\mathbf{X})$ is a test statistic satisfying $0 \leq p(\mathbf{x}) \leq 1$ for every sample point $\mathbf{x}$. Small values of $p(\mathbf{X})$ given evidence that H_1 is true. A *p-value* is *valid* if, for every $\theta \in \Omega_0$ and every $0 \leq \alpha \leq 1$,

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$$\Pr(p(\mathbf{X}) \leq \alpha|\theta) \leq \alpha$$

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  - The smaller the p-value, the stronger, the evidence for rejecting H_0.
  - A p-value reports the results of a test on a more continuous scale
  - Rather than just the dichotomous decision "Accept H_0" or "Reject H_0".

Constructing a value p-value

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$$p(\mathbf{x}) = \sup_{\theta \in \Omega_0} \Pr(W(\mathbf{X}) \geq W(\mathbf{x}) | \theta)$$

Then $p(\mathbf{X})$ is a valid p-value.

Example : Two-sided normal p-value

Problem

Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random sample from a $\mathcal{N}(\theta, \sigma^2)$ population. Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.

1. Construct a size α LRT test.
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Combining the results together

$$\lambda(\mathbf{x}) = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2}$$

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\left| \frac{\overline{x} - \theta_0}{s_{\mathbf{x}} / \sqrt{n}} \right| &\geq c^{***}
\end{aligned}$$

LRT test rejects H_0 if $\left| \frac{\overline{x}-\theta_0}{s_{\mathbf{x}}/\sqrt{n}} \right| \geq c^{***}$. The next step is specify c to get size α test.

Solution - Obtaining size α test

Under H_0

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Therefore, size α LRT test rejects H_0 if and only if $\left|\frac{\bar{x}-\theta_0}{s_{\mathbf{X}}/\sqrt{n}}\right| \geq t_{n-1,\alpha/2}$

Solution - p-value from two-sided test

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$$\begin{aligned} p(\mathbf{x}) &= \sup_{\theta \in \Omega_0} \Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta, \sigma^2) \\ &= \Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta_0, \sigma^2) \\ &= 2\Pr(T_{n-1} \geq W(\mathbf{x})) \\ &= 2\left[1 - F^{-1}_{T_{n-1}}\{W(\mathbf{x})\}\right] \end{aligned}$$

where $F^{-1}_{T_{n-1}}(\cdot)$ is the inverse CDF of t-distribution with $n-1$ degrees of freedom.

Example : One-sided normal p-value

Problem

Let $\mathbf{X} = (X_1, \cdots, X_n)$ be a random sample from a $\mathcal{N}(\theta, \sigma^2)$ population. Consider testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

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$$p(\mathbf{x}) = \sup_{\theta \in \Omega_0} \Pr(W(\mathbf{X}) \geq W(\mathbf{x}) | \theta, \sigma^2)$$

always occurs at when $\theta = \theta_0$, and the value of σ does not matter.

Obtaining one-sided p-value

Consider any $\theta \leq \theta_0$ and any σ.

$$\Pr(W(\mathbf{X}) \geq W(\mathbf{x}) | \theta, \sigma^2) = \Pr\left(\frac{\overline{X} - \theta_0}{s_{\mathbf{X}}/\sqrt{n}} \geq W(\mathbf{x}) | \theta, \sigma^2\right)$$

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&= \Pr\left(\frac{\overline{X} - \theta}{s_{\mathbf{X}}/\sqrt{n}} \geq W(\mathbf{x}) + \frac{\theta_0 - \theta}{s_{\mathbf{X}}/\sqrt{n}}|\theta, \sigma^2\right) \\
&= \Pr\left(T_{n-1} \geq W(\mathbf{x}) + \frac{\theta_0 - \theta}{s_{\mathbf{X}}/\sqrt{n}}|\theta, \sigma^2\right)
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\Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta, \sigma^2) &= \Pr\left(\frac{\overline{X} - \theta_0}{s_{\mathbf{X}}/\sqrt{n}} \geq W(\mathbf{x})|\theta, \sigma^2\right) \\
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&= \Pr\left(T_{n-1} \geq W(\mathbf{x}) + \frac{\theta_0 - \theta}{s_{\mathbf{X}}/\sqrt{n}}|\theta, \sigma^2\right) \\
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Obtaining one-sided p-value

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Suppose $S(\mathbf{X})$ is a sufficient statistic for the model $\{f(\mathbf{x}|\theta) : \theta \in \Omega_0\}$. (not necessarily including alternative hypothesis).

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If we consider only the conditional distribution, by Theorem 8.3.27, this is a valid p-value, meaning that

$$\Pr(p(\mathbf{X}) \leq \alpha | S = s) \leq \alpha$$

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Thus, $p(\mathbf{X})$ is a valid p-value.

Example - Fisher's Exact Test

Problem

Let X_1 and X_2 be independent observations with $X_1 \sim \mathrm{Binomial}(n_1, p_1)$, and $X_2 \sim \mathrm{Binomial}(n_2, p_2)$. Consider testing $H_0 : p_1 = p_2$ versus $H_1 : p_1 > p_2$. Find a valid p-value function.

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Therefore $S = X_1 + X_2$ is a sufficient statistic under H_0.

Solution - Fisher's Exact Test (cont'd)

Given the value of $S = s$, it is reasonable to use X_1 as a test statistic and reject H_0 in favor of H_1 for large values of X_1,because large values of X_1 correspond to small values of $X_2 = s - X_1$.

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$$f(X_1 = x_1|s) = \frac{\binom{n_1}{x_1}\binom{n_2}{s-x_1}}{\binom{n_1+n_2}{s}}$$

Thus, the p-value conditional on the sufficient statistic $s = x_1 + x_2$ is

$$p(x_1, x_2) = \sum_{j=x_1}^{\min(n_1,s)} f(j|s)$$

Exercise 8.1

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1. $H_0 : \theta = 1/2$
2. $H_1 : \theta \neq 1/2$

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Performing size α Hypothesis Testing

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3. Conclude that H_0 is true or false at level α

Obtaining p-value

1. Obtain a p-value function $p(\mathbf{X})$.
2. Compute p-value as a quantitative support for the null hypothesis.

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$$\frac{0.56 - 0.5}{\sqrt{\frac{0.56\times 0.44}{1000}}} = 3.822 > z_{\alpha/2}$$

Hypothesis Testing

Since $z_{\alpha/2}$ is 1.96, 2.57, and 4.42 for $\alpha = 0.05, 0.01$, and 10^{-5}, respectively, we can conclude that the coin is biased at level 0.05 and 0.01. However, at the level of 10^{-5}, the coin can be assumed to be fair.

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So, under the null hypothesis, the size of test is less than 1.32×10^{-4}, suggesting a strong evidence for rejecting H_0.

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Solution - Hypothesis

$X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$.

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Solution - Hypothesis

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Constructing a test based on sufficient statistic (cont'd)

The conditional distribution of **x** given s is

$$f(x_1, x_2 | s) = \frac{f_{\mathbf{x}}(x_1, x_2 | \lambda)}{f_S(s | \lambda)}$$

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where $x_1 = 15$, $x_2 = 10$.

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where $x_1 = 15$, $x_2 = 10$. Therefore, H_0 is not rejected when $\alpha < .05$, and it is not reasonable to claim that the accident rate has dropped.

Summary

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- p-Value
- Fisher's Exact Test
- Examples of Hypothesis Testing

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Next Lectures

- Interval Estimation
- Confidence Interval