Biostatistics 602 - Statistical Inference
Lecture 05
Complete Statistics

Hyun Min Kang

January 24th, 2013

1. What is an ancillary statistic for θ?

1. What is an ancillary statistic for θ?
2. Can an ancillary statistic be a sufficient statistic?

1. What is an ancillary statistic for θ?
2. Can an ancillary statistic be a sufficient statistic?
3. What are the location parameter and the scale parameter?

1. What is an ancillary statistic for θ?
2. Can an ancillary statistic be a sufficient statistic?
3. What are the location parameter and the scale parameter?
4. In which case ancillary statistics would be helpful?

Last Lecture : Ancillary Statistics

Definition 6.2.16

A statistic $S(\mathbf{X})$ is an *ancillary statistic* if its distribution does not depend on θ.

Last Lecture : Ancillary Statistics

Definition 6.2.16

A statistic $S(\mathbf{X})$ is an *ancillary statistic* if its distribution does not depend on θ.

Examples of Ancillary Statistics

- $X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where σ^2 is known.
- $s^2_{\mathbf{X}} = \frac{1}{n-1} \sum_{i=1}^n (X_1 - \overline{X})^2$ is an ancillary statistic
- $X_1 - X_2 \sim \mathcal{N}(0, 2\sigma^2)$ is ancillary.
- $(X_1 + X_2)/2 - X_3 \sim \mathcal{N}(0, 1.5\sigma^2)$ is ancillary.
- $\frac{(n-1)s^2_{\mathbf{X}}}{\sigma^2} \sim \chi^2_{n-1}$ is ancillary.

Example : Uniform Ancillary Statistics

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1)$.
- Show that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic.

Example : Uniform Ancillary Statistics

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1)$.
- Show that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic.

Possible Strategies

- Method 1 : Obtain the distribution of R and show that it is independent of θ.

Example : Uniform Ancillary Statistics

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1)$.
- Show that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic.

Possible Strategies

- Method 1 : Obtain the distribution of R and show that it is independent of θ.
- Method 2 : Represent R as a function of ancillary statistics, which is independent of θ.

Method 2 - A Simpler Proof

$$f_X(x|\theta) = I(\theta < x < \theta + 1) = I(0 < x - \theta < 1)$$

Let $Y_i = X_i - \theta \sim \text{Uniform}(0, 1)$. Then $X_i = Y_i + \theta$, $|\frac{dx}{dy}| = 1$ holds.

$$f_Y(y) = I(0 < y + \theta - \theta < 1)|\frac{dx}{dy}| = I(0 < y < 1)$$

Method 2 - A Simpler Proof

$$f_X(x|\theta) = I(\theta < x < \theta + 1) = I(0 < x - \theta < 1)$$

Let $Y_i = X_i - \theta \sim \text{Uniform}(0,1)$. Then $X_i = Y_i + \theta$, $|\frac{dx}{dy}| = 1$ holds.

$$f_Y(y) = I(0 < y + \theta - \theta < 1)|\frac{dx}{dy}| = I(0 < y < 1)$$

Then, the range statistic R can be rewritten as follows.

$$R = X_{(n)} - X_{(1)} = (Y_{(n)} + \theta) - (Y_{(1)} + \theta) = Y_{(n)} - Y_{(1)}$$

Method 2 - A Simpler Proof

$$f_X(x|\theta) = I(\theta < x < \theta + 1) = I(0 < x - \theta < 1)$$

Let $Y_i = X_i - \theta \sim \text{Uniform}(0,1)$. Then $X_i = Y_i + \theta$, $|\frac{dx}{dy}| = 1$ holds.

$$f_Y(y) = I(0 < y + \theta - \theta < 1)|\frac{dx}{dy}| = I(0 < y < 1)$$

Then, the range statistic R can be rewritten as follows.

$$R = X_{(n)} - X_{(1)} = (Y_{(n)} + \theta) - (Y_{(1)} + \theta) = Y_{(n)} - Y_{(1)}$$

As $Y_{(n)} - Y_{(1)}$ is a function of $Y_1, \cdots, Y_n$. Any joint distribution of $Y_1, \cdots, Y_n$ does not depend on θ. Therefore, R is an ancillary statistic.

Location-Scale Family and Parameters

Definition 3.5.5

Let $f(x)$ be any pdf. Then for any $\mu, -\infty < \mu < \infty$, and any $\sigma > 0$ the family of pdfs $f((x-\mu)/\sigma)/\sigma$, indexed by the parameter (μ, σ) is called the *location-scale family with standard pdf* $f(x)$, and μ is called the *location parameter* and σ is called the *scale parameter* for the family.

Location-Scale Family and Parameters

Definition 3.5.5

Let $f(x)$ be any pdf. Then for any $\mu, -\infty < \mu < \infty$, and any $\sigma > 0$ the family of pdfs $f((x-\mu)/\sigma)/\sigma$, indexed by the parameter (μ, σ) is called the *location-scale family with standard pdf* $f(x)$, and μ is called the *location parameter* and σ is called the *scale parameter* for the family.

Example

- $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sim \mathcal{N}(0,1)$
- $f((x-\mu)/\sigma)/\sigma = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \sim \mathcal{N}(\mu, \sigma^2)$

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
    - In other words, $g(T) = 0$ almost surely.

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
  - In other words, $g(T) = 0$ almost surely.
- Equivalently, $T(\mathbf{X})$ is called a *complete statistic*

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
  - In other words, $g(T) = 0$ almost surely.
- Equivalently, $T(\mathbf{X})$ is called a *complete statistic*

Example

- $T(X) \sim \mathcal{N}(0, 1)$
- $g_1(T(\mathbf{X})) = 0 \Longrightarrow \Pr[g_1(T(\mathbf{X})) = 0] = 1.$

Complete Statistics

Definition

- Let $\mathcal{T} = \{f_T(t|\theta), \theta \in \Omega\}$ be a family of pdfs or pmfs for a statistic $T(\mathbf{X})$.
- The family of probability distributions is called *complete* if
- $E[g(T)|\theta] = 0$ for all θ implies $\Pr[g(T) = 0|\theta] = 1$ for all θ.
  - In other words, $g(T) = 0$ almost surely.
- Equivalently, $T(\mathbf{X})$ is called a *complete statistic*

Example

- $T(X) \sim \mathcal{N}(0, 1)$
- $g_1(T(\mathbf{X})) = 0 \implies \Pr[g_1(T(\mathbf{X})) = 0] = 1$.
- $g_2(T(\mathbf{X})) = I(T(\mathbf{X}) = 0) \implies \Pr[g_2(T(\mathbf{X})) = 0] = 1 - \Pr[T(\mathbf{X}) = 0)]$. In this case, $g_2(T(\mathbf{X})) = 0$ is almost surely true.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.
- For example, $X \sim \mathcal{N}(0,1)$ and $g(x) = x$ makes $E[g(\mathbf{X})] = E\mathbf{X} = 0$, but $\Pr(g(\mathbf{X}) = 0) = 0$ instead of 1.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.
- For example, $X \sim \mathcal{N}(0,1)$ and $g(x) = x$ makes $E[g(\mathbf{X})] = E\mathbf{X} = 0$, but $\Pr(g(\mathbf{X}) = 0) = 0$ instead of 1.
- The above example is only for a particular distribution, not a family of distributions.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.
- For example, $X \sim \mathcal{N}(0,1)$ and $g(x) = x$ makes $E[g(\mathbf{X})] = E\mathbf{X} = 0$, but $\Pr(g(\mathbf{X}) = 0) = 0$ instead of 1.
- The above example is only for a particular distribution, not a family of distributions.
- If $X \sim \mathcal{N}(\theta, 1), -\infty < \theta < \infty$, then no function of X except for $g(\mathbf{X}) = 0$ satisfies $E[g(\mathbf{X})|\theta]$ for all θ.

Notes on Complete Statistics

- Notice that completeness is a property of a family of probability distributions, not of a particular distribution.
- For example, $X \sim \mathcal{N}(0,1)$ and $g(x) = x$ makes $E[g(\mathbf{X})] = E\mathbf{X} = 0$, but $\Pr(g(\mathbf{X}) = 0) = 0$ instead of 1.
- The above example is only for a particular distribution, not a family of distributions.
- If $X \sim \mathcal{N}(\theta, 1), -\infty < \theta < \infty$, then no function of X except for $g(\mathbf{X}) = 0$ satisfies $E[g(\mathbf{X})|\theta]$ for all θ.
- Therefore, the family of $\mathcal{N}(\theta, 1)$ distributions, $-\infty < \theta < \infty$, is complete.

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness,

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name,

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete",

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.
- Requiring $g(T)$ to satisfy the definition puts a restriction on g.

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.
- Requiring $g(T)$ to satisfy the definition puts a restriction on g. The larger the family of pdfs/pmfs, the greater the restriction on g.

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.
- Requiring $g(T)$ to satisfy the definition puts a restriction on g. The larger the family of pdfs/pmfs, the greater the restriction on g. When the family of pdfs/pmfs is augmented to the point that $E[g(T)] = 0$ for all θ,

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.
- Requiring $g(T)$ to satisfy the definition puts a restriction on g. The larger the family of pdfs/pmfs, the greater the restriction on g. When the family of pdfs/pmfs is augmented to the point that $E[g(T)] = 0$ for all θ, it rules out all g except for the trivial $g(T) = 0$,

Why "Complete" Statistics?

Stigler (1972) Am. Stat. 26(2):28-9

- While a student encountering completeness for the first time is very likely to appreciate its usefulness, he is just as likely to be puzzled by its name, and wonder what connection (if any) there is between the statistical use of the term "complete", and the dictionary definition: lacking none of the parts, whole, entire.
- Requiring $g(T)$ to satisfy the definition puts a restriction on g. The larger the family of pdfs/pmfs, the greater the restriction on g. When the family of pdfs/pmfs is augmented to the point that $E[g(T)] = 0$ for all θ, it rules out all g except for the trivial $g(T) = 0$, then the family is said to be complete.

Example - Poisson distribution

Problem

- Suppose $\mathcal{T} = \left\{ f_T : f_T(t|\lambda) = \frac{\lambda^t e^{-\lambda}}{t!} \right\}$ for $t \in \{0, 1, 2, \cdots\}$. Let $\lambda \in \Omega = \{1, 2\}$. Show that this family is NOT complete

Example - Poisson distribution

Problem

- Suppose $\mathcal{T} = \left\{ f_T : f_T(t|\lambda) = \frac{\lambda^t e^{-\lambda}}{t!} \right\}$ for $t \in \{0, 1, 2, \cdots\}$. Let $\lambda \in \Omega = \{1, 2\}$. Show that this family is NOT complete

Proof strategy

- We need to find a counter example,

Example - Poisson distribution

Problem

- Suppose $\mathcal{T} = \left\{ f_T : f_T(t|\lambda) = \frac{\lambda^t e^{-\lambda}}{t!} \right\}$ for $t \in \{0, 1, 2, \cdots\}$. Let $\lambda \in \Omega = \{1, 2\}$. Show that this family is NOT complete

Proof strategy

- We need to find a counter example,
- which is a function g such that $E[g(T)|\lambda] = 0$ for $\lambda = 1, 2$ but $g(T) \neq 0$.

Poisson distribution example : Proof

The function g must satisfy

$$E[g(T)|\lambda] = \sum_{t=0}^{\infty} g(t) \frac{\lambda^t e^{-\lambda}}{t!} = 0$$

Poisson distribution example : Proof

The function g must satisfy

$$E[g(T)|\lambda] = \sum_{t=0}^{\infty} g(t) \frac{\lambda^t e^{-\lambda}}{t!} = 0$$

for $\lambda \in \{1, 2\}$. Thus,

$$\begin{cases} E[g(T)|\lambda = 1] & = & \sum_{t=0}^{\infty} g(t) \frac{1^t e^{-1}}{t!} = 0 \\ E[g(T)|\lambda = 2] & = & \sum_{t=0}^{\infty} g(t) \frac{2^t e^{-2}}{t!} = 0 \end{cases}$$

Poisson distribution example : Proof

The function g must satisfy

$$E[g(T)|\lambda] = \sum_{t=0}^{\infty} g(t) \frac{\lambda^t e^{-\lambda}}{t!} = 0$$

for $\lambda \in \{1, 2\}$. Thus,

$$\begin{cases} E[g(T)|\lambda = 1] & = & \sum_{t=0}^{\infty} g(t) \frac{1^t e^{-1}}{t!} = 0 \\ E[g(T)|\lambda = 2] & = & \sum_{t=0}^{\infty} g(t) \frac{2^t e^{-2}}{t!} = 0 \end{cases}$$

The above equation can be rewritten as

$$\begin{cases} \sum_{t=0}^{\infty} g(t)/t! & = & 0 \\ \sum_{t=0}^{\infty} 2^t g(t)/t! & = & 0 \end{cases}$$

Poisson distribution example : Proof (cont'd)

Define $g(t)$ as

$$g(t) = \begin{cases} 2 & t = 0 \lor t = 2 \\ -3 & t = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\sum_{t=0}^{\infty} g(t)/t! = g(0)/0! + g(1)/1! + g(2)/2! = 2 - 3 + 2/2 = 0$$

$$\sum_{t=0}^{\infty} 2^t g(t)/t! = g(0)/0! + 2g(1)/1! + 2^2 g(2)/2! = 2 - 6 + 8/2 = 0$$

There exists a non-zero function g that satisfies $E[g(T)\lambda] = 0$ for all $\lambda \in \Omega$. Therefore this family is NOT complete.

Another example with Poisson distribution

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda), \lambda > 0.$
- Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a complete statistic.

Another example with Poisson distribution

Problem

- $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda), \lambda > 0.$
- Show that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a complete statistic.

Proof strategy

- Need to find the distribution of $T(\mathbf{X})$
- Show that there is no non-zero function g such that $E[g(T)|\lambda] = 0$ for all λ.

Proof : Finding the moment-generating function of X

$$M_X(s) = E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!}$$

Proof : Finding the moment-generating function of X

$$\begin{aligned} M_X(s) &= E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^s\lambda)^x}{x!} e^{-e^s\lambda} e^{e^s\lambda} \end{aligned}$$

Proof : Finding the moment-generating function of X

$$
\begin{aligned}
M_X(s) &= E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^s\lambda)^x}{x!} e^{-e^s\lambda} e^{e^s\lambda} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} e^{e^s\lambda} \frac{(e^s\lambda)^x e^{-e^s\lambda}}{x!}
\end{aligned}
$$

Proof : Finding the moment-generating function of X

$$
\begin{aligned}
M_X(s) &= E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^s\lambda)^x}{x!} e^{-e^s\lambda} e^{e^s\lambda} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} e^{e^s\lambda} \frac{(e^s\lambda)^x e^{-e^s\lambda}}{x!} \\
&= e^{\lambda} e^{e^s\lambda} \sum_{x=0}^{\infty} f_{Poisson}(x|e^s\lambda)
\end{aligned}
$$

Proof : Finding the moment-generating function of X

$$\begin{aligned}
M_X(s) &= E[e^{sX}] = \sum_{x=0}^{\infty} e^{sx} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(e^s\lambda)^x}{x!} e^{-e^s\lambda} e^{e^s\lambda} \\
&= \sum_{x=0}^{\infty} e^{-\lambda} e^{e^s\lambda} \frac{(e^s\lambda)^x e^{-e^s\lambda}}{x!} \\
&= e^{\lambda} e^{e^s\lambda} \sum_{x=0}^{\infty} f_{Poisson}(x|e^s\lambda) \\
&= e^{\lambda(e^s-1)}
\end{aligned}$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$

$$M_T(s) = E(e^{s \sum X_i}) =$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$

$$M_T(s) = E(e^{s \sum X_i}) = E\left(\prod_{i=1}^{n} e^{sX_i}\right)$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$

$$\begin{aligned} M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^{n} e^{sX_i}\right) \\ &= \prod_{i=1}^{n} E\left(e^{sX_i}\right) = \end{aligned}$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$\begin{aligned} M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i}\right) \\ &= \prod_{i=1}^n E\left(e^{sX_i}\right) = \left[E\left(e^{sX_i}\right)\right]^n \end{aligned}$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$
\begin{aligned}
M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i}\right) \\
&= \prod_{i=1}^n E\left(e^{sX_i}\right) = \left[E\left(e^{sX_i}\right)\right]^n \\
&= \left[e^{-\lambda(e^s-1)}\right]^n =
\end{aligned}
$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$
\begin{aligned}
M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i}\right) \\
&= \prod_{i=1}^n E\left(e^{sX_i}\right) = \left[E\left(e^{sX_i}\right)\right]^n \\
&= \left[e^{-\lambda(e^s-1)}\right]^n = e^{n\lambda(e^s-1)}
\end{aligned}
$$

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$
\begin{aligned}
M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i}\right) \\
&= \prod_{i=1}^n E\left(e^{sX_i}\right) = \left[E\left(e^{sX_i}\right)\right]^n \\
&= \left[e^{-\lambda(e^s-1)}\right]^n = e^{n\lambda(e^s-1)}
\end{aligned}
$$

Theorem 2.3.11 (b)

Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exists. If the moment generating functions exists and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

Proof : Finding the MGF of $T(\mathbf{X}) = \sum_{i=1}^n X_i$

$$\begin{aligned} M_T(s) &= E(e^{s\sum X_i}) = E\left(\prod_{i=1}^n e^{sX_i}\right) \\ &= \prod_{i=1}^n E\left(e^{sX_i}\right) = \left[E\left(e^{sX_i}\right)\right]^n \\ &= \left[e^{-\lambda(e^s-1)}\right]^n = e^{n\lambda(e^s-1)} \end{aligned}$$

Theorem 2.3.11 (b)

Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exists. If the moment generating functions exists and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.

By Theorem 2.3.11, $T(\mathbf{X}) \sim \text{Poisson}(n\lambda)$.

Proof : Showing $E[g(T)|\lambda] = 0 \Longleftrightarrow \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

Proof : Showing $E[g(T)|\lambda] = 0 \Longleftrightarrow \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$E[g(T)|\lambda] = \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!}$$

Proof : Showing $E[g(T)|\lambda] = 0 \iff \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$\begin{aligned} E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\ &= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0 \end{aligned}$$

Proof : Showing $E[g(T)|\lambda] = 0 \Longleftrightarrow \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$\begin{aligned} E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\ &= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0 \end{aligned}$$

Which is equivalent to

$$\sum_{t=0}^{\infty} \frac{g(t)n^t}{t!}\lambda^t = 0$$

Proof : Showing $E[g(T)|\lambda] = 0 \iff \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$\begin{aligned} E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\ &= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0 \end{aligned}$$

Which is equivalent to

$$\sum_{t=0}^{\infty} \frac{g(t)n^t}{t!} \lambda^t = 0$$

for all $\lambda > 0$.

Proof : Showing $E[g(T)|\lambda] = 0 \iff \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$\begin{aligned} E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\ &= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0 \end{aligned}$$

Which is equivalent to

$$\sum_{t=0}^{\infty} \frac{g(t)n^t}{t!} \lambda^t = 0$$

for all $\lambda > 0$. Because the function above is a power series expansion of λ, $g(t)n^t/t! = 0$ for all t. and $g(t) = 0$ for all t.

Proof : Showing $E[g(T)|\lambda] = 0 \iff \Pr[g(T) = 0] = 1$

. Suppose that there exists a $g(T)$ such that $E[g(T)|\lambda] = 0$ for all $\lambda > 0$.

$$\begin{aligned} E[g(T)|\lambda] &= \sum_{t=0}^{\infty} \frac{e^{-n\lambda}(n\lambda)^t}{t!} \\ &= e^{-n\lambda} \sum_{t=0}^{\infty} \frac{g(t)(n\lambda)^t}{t!} = 0 \end{aligned}$$

Which is equivalent to

$$\sum_{t=0}^{\infty} \frac{g(t)n^t}{t!}\lambda^t = 0$$

for all $\lambda > 0$. Because the function above is a power series expansion of λ, $g(t)n^t/t! = 0$ for all t. and $g(t) = 0$ for all t. Therefore $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a complete statistic.

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$.

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is $F_X(x) = \frac{x}{\theta} I(0 < x < \theta) + I(x \geq \theta)$.

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is $F_X(x) = \frac{x}{\theta} I(0 < x < \theta) + I(x \geq \theta)$.

$$f_T(t|\theta) = \frac{n!}{(n-1)!} f_X(t) F_X(t)^{n-1}$$

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is $F_X(x) = \frac{x}{\theta} I(0 < x < \theta) + I(x \geq \theta)$.

$$
\begin{aligned}
f_T(t|\theta) &= \frac{n!}{(n-1)!} f_X(t) F_X(t)^{n-1} \\
&= \frac{n}{\theta} \left(\frac{t}{\theta}\right)^{n-1} I(0 < t < \theta)
\end{aligned}
$$

Example : Uniform Distribution

Problem

Let $X_1, \cdots, X_n \stackrel{\text{i.i.d.}}{\sim} \text{Uniform}(0, \theta),\ \theta > 0,\ \Omega = (0, \infty)$.
Show that $X_{(n)}$ is complete.

Proof

We need to obtain the distribution of $T(\mathbf{X}) = X_{(n)}$. Let $f_X(x) = \frac{1}{\theta} I(0 < x < \theta)$, then its cdf is $F_X(x) = \frac{x}{\theta} I(0 < x < \theta) + I(x \geq \theta)$.

$$\begin{aligned} f_T(t|\theta) &= \frac{n!}{(n-1)!} f_X(t) F_X(t)^{n-1} \\ &= \frac{n}{\theta} \left(\frac{t}{\theta}\right)^{n-1} I(0 < t < \theta) = n\theta^{-n} t^{n-1} I(0 < t < \theta) \end{aligned}$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$E[g(T)|\theta] = \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta)\, dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1}\, dt = 0 \end{aligned}$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \end{aligned}$$

Taking derivative of both sides,

$$\frac{n}{\theta^n} g(\theta)\theta^{n-1} - \frac{n^2}{\theta^{n+1}} \int_0^\theta g(t) t^{n-1} dt = 0$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$E[g(T)|\theta] = \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt$$
$$= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0$$

Taking derivative of both sides,

$$\frac{n}{\theta^n} g(\theta)\theta^{n-1} - \frac{n^2}{\theta^{n+1}} \int_0^\theta g(t) t^{n-1} dt = 0$$
$$\frac{ng(\theta)}{\theta} = \frac{n}{\theta}\frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = \frac{n}{\theta} E[g(T)|\theta] = 0$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \end{aligned}$$

Taking derivative of both sides,

$$\frac{n}{\theta^n} g(\theta)\theta^{n-1} - \frac{n^2}{\theta^{n+1}} \int_0^\theta g(t) t^{n-1} dt = 0$$

$$\frac{ng(\theta)}{\theta} = \frac{n}{\theta}\frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = \frac{n}{\theta} E[g(T)|\theta] = 0$$

Proof : Uniform Distribution (cont'd)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$E[g(T)|\theta] = \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt$$
$$= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0$$

Taking derivative of both sides,

$$\frac{n}{\theta^n} g(\theta)\theta^{n-1} - \frac{n^2}{\theta^{n+1}} \int_0^\theta g(t) t^{n-1} dt = 0$$
$$\frac{ng(\theta)}{\theta} = \frac{n}{\theta}\frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = \frac{n}{\theta} E[g(T)|\theta] = 0$$

Because $g(T) = 0$ holds for all $\theta > 0$, $T(\mathbf{X}) = X_{(n)}$ is a complete statistic.

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$E[g(T)|\theta] = \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt$$

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \end{aligned}$$

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \\ \int_0^\theta g(t) t^{n-1} dt &= 0 \end{aligned}$$

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$
\begin{aligned}
E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\
&= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \\
\int_0^\theta g(t) t^{n-1} dt &= 0
\end{aligned}
$$

Taking derivative of both sides,

$$
\begin{aligned}
g(\theta)\theta^{n-1} &= 0 \\
g(\theta) &= 0
\end{aligned}
$$

A simpler proof (how it was solved in the class)

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta > 0$

$$\begin{aligned} E[g(T)|\theta] &= \int_0^\theta g(t) n\theta^{-n} t^{n-1} I(0 < t < \theta) dt \\ &= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt = 0 \\ \int_0^\theta g(t) t^{n-1} dt &= 0 \end{aligned}$$

Taking derivative of both sides,

$$\begin{aligned} g(\theta)\theta^{n-1} &= 0 \\ g(\theta) &= 0 \end{aligned}$$

for all $\theta > 0$. Because $g(T) = 0$ holds for all $\theta > 0$, $T(\mathbf{X}) = X_{(n)}$ is a complete statistic.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta + 1),\ \theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.
- Show that $\mathbf{T}(\mathbf{X})$ is not a complete statistic.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.
- Show that $\mathbf{T}(\mathbf{X})$ is not a complete statistic.

Proof - Using a range statistic

Define $R = X_{(n)} - X_{(1)}$.

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.
- Show that $\mathbf{T}(\mathbf{X})$ is not a complete statistic.

Proof - Using a range statistic

Define $R = X_{(n)} - X_{(1)}$. We have previously shown that

$$f_R(r|\theta) = n(n-1)r^{(n-2)}(1-r) \qquad ,0 < r < 1$$

Another Example of Uniform Distribution

Problem

- Let $X_1, \cdots, X_n \overset{\text{i.i.d.}}{\sim} \text{Uniform}(\theta, \theta+1),\ \theta \in \mathbb{R}$.
- We have previously shown that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.
- Show that $\mathbf{T}(\mathbf{X})$ is not a complete statistic.

Proof - Using a range statistic

Define $R = X_{(n)} - X_{(1)}$. We have previously shown that

$$f_R(r|\theta) = n(n-1)r^{(n-2)}(1-r) \qquad ,0 < r < 1$$

Then $R \sim \text{Beta}(n-1, 2)$, and $E[R|\theta] = \frac{n-1}{n+1}$.

Proof

Define $g(\mathbf{T}(X)) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$

Proof

Define $g(\mathbf{T}(X)) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$

$$E[g(\mathbf{T})|\theta] = E[X_{(n)} - X_{(1)}|\theta] - \frac{n-1}{n+1}$$

Proof

Define $g(\mathbf{T}(X)) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$

$$\begin{aligned} E[g(\mathbf{T})|\theta] &= E[X_{(n)} - X_{(1)}|\theta] - \frac{n-1}{n+1} \\ &= \frac{n-1}{n+1} - \frac{n-1}{n+1} = 0 \end{aligned}$$

Therefore, there exist a $g(\mathbf{T})$ such that $\Pr[g(\mathbf{T})|\theta] < 1$ for all θ, so $\mathbf{T}(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is not a complete statistic.

Even Simpler Proof

- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.

Even Simpler Proof

- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.
- Define $g(\mathbf{T}) = X_{(n)} - X_{(1)} - E(R)$. Note that $E(R)$ is constant to θ.

Even Simpler Proof

- We know that $R = X_{(n)} - X_{(1)}$ is an ancillary statistic, which do not depend on θ.
- Define $g(\mathbf{T}) = X_{(n)} - X_{(1)} - E(R)$. Note that $E(R)$ is constant to θ.
- Then $E[g(\mathbf{T})|\theta] = E(R) - E(R) = 0$, so T is not a complete statistic.

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$.

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$. Is $T(X) = X$ a complete statistic?

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$. Is $T(X) = X$ a complete statistic?

Solution

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta \in \mathbb{N}$.
Note that $f_X(x) = \frac{1}{\theta} I(x \in \{1, \cdots, \theta\}) = \frac{1}{\theta} I_{\mathbb{N}_\theta}(x)$.

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$. Is $T(X) = X$ a complete statistic?

Solution

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta \in \mathbb{N}$.
Note that $f_X(x) = \frac{1}{\theta} I(x \in \{1, \cdots, \theta\}) = \frac{1}{\theta} I_{\mathbb{N}_\theta}(x)$.

$$E[g(T)|\theta] = E[g(X)|\theta] = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0$$

Example from Stigler (1972) Am. Stat.

Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N}$. Is $T(X) = X$ a complete statistic?

Solution

Consider a function $g(T)$ such that $E[g(T)|\theta] = 0$ for all $\theta \in \mathbb{N}$.
Note that $f_X(x) = \frac{1}{\theta} I(x \in \{1, \cdots, \theta\}) = \frac{1}{\theta} I_{\mathbb{N}_\theta}(x)$.

$$
\begin{aligned}
E[g(T)|\theta] &= E[g(X)|\theta] = \sum_{x=1}^{\theta} \frac{1}{\theta} g(x) = \frac{1}{\theta} \sum_{x=1}^{\theta} g(x) = 0 \\
\sum_{x=1}^{\theta} g(x) &= 0
\end{aligned}
$$

Solution (cont'd)

for all $\theta \in \mathbb{N}$, which implies

- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$

Solution (cont'd)

for all $\theta \in \mathbb{N}$, which implies

- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$
- if $\theta = 2$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = g(2) = 0$.

Solution (cont'd)

for all $\theta \in \mathbb{N}$, which implies

- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$
- if $\theta = 2$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = g(2) = 0$.
- $\vdots$
- if $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + \cdots + g(k-1) + g(2) = g(k) = 0$.

Solution (cont'd)

for all $\theta \in \mathbb{N}$, which implies

- if $\theta = 1$, $\sum_{x=1}^{\theta} g(x) = g(1) = 0$
- if $\theta = 2$, $\sum_{x=1}^{\theta} g(x) = g(1) + g(2) = g(2) = 0$.
- $\vdots$
- if $\theta = k$, $\sum_{x=1}^{\theta} g(x) = g(1) + \cdots + g(k-1) + g(2) = g(k) = 0$.

Therefore, $g(x) = 0$ for all $x \in \mathbb{N}$, and $T(X) = X$ is a complete statistic for $\theta \in \Omega = \mathbb{N}$.

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$.

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$. Is $T(X) = X$ a complete statistic?

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$. Is $T(X) = X$ a complete statistic?

Solution

Define a nonzero $g(x)$ as follows

$$g(x) = \begin{cases} 1 & x = n \\ -1 & x = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$. Is $T(X) = X$ a complete statistic?

Solution

Define a nonzero $g(x)$ as follows

$$g(x) = \begin{cases} 1 & x = n \\ -1 & x = n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[g(T)|\theta] = \frac{1}{\theta}\sum_{x=1}^{\theta} g(x) = \begin{cases} 0 & \theta \neq n \\ \frac{1}{\theta} & \theta = n \end{cases}$$

Is the previous example barely complete?

Modified Problem

Let X is a uniform random sample from $\{1, \cdots, \theta\}$ where $\theta \in \Omega = \mathbb{N} - \{n\}$. Is $T(X) = X$ a complete statistic?

Solution

Define a nonzero $g(x)$ as follows

$$g(x) = \begin{cases} 1 & x = n \\ -1 & x = n+1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[g(T)|\theta] = \frac{1}{\theta}\sum_{x=1}^{\theta} g(x) = \begin{cases} 0 & \theta \neq n \\ \frac{1}{\theta} & \theta = n \end{cases}$$

Because Ω does not include n, $g(x) = 0$ for all $\theta \in \Omega = \mathbb{N} - \{n\}$, and $T(X) = X$ is not a complete statistic.

Summary

Today - Complete Statistics

- Examples of complete statistics
- Two Poisson distribution examples
- Two Uniform distribution examples
- Example of barely complete statistics.

Summary

Today - Complete Statistics

- Examples of complete statistics
- Two Poisson distribution examples
- Two Uniform distribution examples
- Example of barely complete statistics.

Next Lecture

- Basu's Theorem