Biostatistics 615/815 Lecture 5: Divide and Conquer Algorithms Sorting Algorithms

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Recap - An example C++ class

```
#include <iostream>
#include <cmath>
class Point {
public:
   double x;
   double y;
   // A constructor can call constructor of each member variable
   Point(double px, double py) : px(x), py(y) {}
   // equivalent to -- Point(double px, double py) : x(px), y(py) {}
   double distanceFromOrigin() { return sqrt( x*x + y*y );}
};
int main(int argc, char** argv) {
  Point p(3,4); // calls constructor with two arguments
  std::cout << p.distanceFromOrigin() << std::endl; // prints 5
}
```

Recap: STL in pratice

sortedEcho.cpp

```
#include <iostream>
#include <string>
#include <vector>
int main(int argc, char** argv) {
  std::vector<std::string> vArgs; // vector of strings
  for(int i=1; i < argc; ++i) {
    vArgs.push_back(argv[i]);  // append each arguments to the vector
  }
  std::sort(vArgs.begin(),vArgs.end()); // sort the vector in alphanumeric order
  std::cout << "Sorted arguments :";    // print the sorted arguments
  for(int i=0; i < vArgs.size(); ++i) { std::cout << " " << vArgs[i]; }
  std::cout << std::endl;
  return 0;
}
```

A running example

```
user@host:~/> ./sortedEcho Hello, World! hello, world! 2 3 5 60 1
Sorted arguments : 1 2 3 5 60 Hello, World! hello, world!
```

Recap: Euclid's algorithm

Algorithm GCD

Data: Two integers a and b

Result: The greatest common divisor (GCD) between a and b

if a *divides* b **then**

  return *a*

else

  Find the largest integer t such that $at + r = b$;

  return $\textsc{Gcd}(r,a)$

end

Recap: Euclid's algorithm

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Function gcd()

```
int gcd (int a, int b) {
  if ( a == 0 )  return b; // equivalent to returning a when b % a == 0
  else           return gcd( b % a, a );
}
```

Divide-and-conquer algorithms

Solve a problem recursively, applying three steps at each level of recursion

- **Divide** the problem into a number of subproblems that are smaller instances of the same problem
- **Conquer** the subproblems by solving them recursively. If the subproblem sizes are small enough, however, just solve the subproblems in a straightforward manner.
- **Combine** the solutions to subproblems into the solution for the original problem

Binary Search

```
// assuming a is sorted, return index of array containing the key,
// among a[start...end]. Return -1 if no key is found
int binarySearch(std::vector<int>& a, int key, int start, int end) {
  if ( start > end ) return -1; // search failed
  int mid = (start+end)/2;
  if ( key == a[mid] ) return mid; // terminate if match is found
  if ( key < a[mid] ) // divide the remaining problem into half
    return binarySearch(a, key, start, mid-1);
  else
    return binarySearch(a, key, mid+1, end);
}
```

Recursive Maximum

```
// find maximum within an a[start..end]
int findMax(std::vector<int>& a, int start, int end) {
  if ( start == end ) return a[start];  // conquer small problem directly
  else {
    int mid = (start+end)/2;
    int leftMax = findMax(a,start,mid);  // divide the problem into half
    int rightMax = findMax(a,mid+1,end);
    return ( leftMax > rightMax ? leftMax : rightMax ); // combine solutions
  }
}
```

Reading from Files : stdSort.cpp

```
#include <iostream>
#include <fstream>
#include <vector>
int main(int argc, char** argv) { // sorting software using std::sort
  int tok;
  std::vector<int> v;
  if ( argc > 1 ) { // if argument is given, read from file
    std::ifstream fin(argv[1]);
    while( fin >> tok ) { v.push_back(tok); }
    fin.close();
  }
  else { // read from standard input if no argument is specified
    while( std::cin >> tok ) { v.push_back(tok); }
  }
  std::sort(v.begin(), v.end()); // Sort using the algorithm in STL
  for(int i=0; i < v.size(); ++i) {
    std::cout << v[i] << std::endl; // print out the content
  }
  return 0;
}
```

Reading from Files : insertionSort.cpp

```
#include <iostream>
#include <fstream>
#include <vector>
void insertionSort(std::vector<int>& v); // insertionSort as defined before
int main(int argc, char** argv) {
  int tok;
  std::vector<int> v;
  if ( argc > 1 ) {
    std::ifstream fin(argv[1]);
    while( fin >> tok ) { v.push_back(tok); }
    fin.close();
  }
  else {
    while( std::cin >> tok ) { v.push_back(tok); }
  }
  insertionSort(v); // differs from stdSort in only this part
  for(int i=0; i < v.size(); ++i) {
    std::cout << v[i] << std::endl;
  }
  return 0;
}
```

STL Use in INSERTIONSORT Algorithm

insertionSort.cpp - insertionSort() function

```
// perform insertion sort on A
void insertionSort(std::vector<int>& A) { // call-by-reference
  for(int j=1; j < A.size(); ++j) { // 0-based index
    int key = A[j];  // key element to relocate
    int i = j-1;     // index to be relocated
    while( (i >= 0) && (A[i] > key) ) { // find position to relocate
      A[i+1] = A[i]; // shift elements
      --i;           // update index to be relocated
    }
    A[i+1] = key;    // relocate the key element
  }
}
```

Running time comparison

Running example with 100,000 elements (in UNIX or MacOS)

```
user@host:~/> time cat src/sample.input.txt | src/stdSort > /dev/null
real 0m0.430s
user 0m0.281s
sys 0m0.130s

user@host:~/> time cat src/sample.input.txt | src/insertionSort > /dev/null
real 1m8.795s
user 1m8.181s
sys 0m0.206s
```

Merge Sort

Divide and conquer algorithm

Divide Divide the n element sequence to be sorted into two subsequences of n/2 elements each

Conquer Sort the two subsequences recursively using merge sort

Combine Merge the two sorted subsequences to produce the sorted answer

mergeSort.cpp - main()

```
#include <iostream>
#include <vector>
#include <climits>
void mergeSort(std::vector<int>& a, int p, int r); // defined later
void printArray(std::vector<int>& A); // same as insertionSort
// same to insertionSort.cpp except for one line
int main(int argc, char** argv) {
  std::vector<int> v;
  int tok;
  while ( std::cin >> tok ) {
    v.push_back(tok);
  }
  std::cout << "Before sorting: ";
  printArray(v);
  mergeSort(v, 0, v.size()-1); // differs from insertionSort.cpp
  std::cout << "After sorting: ";
  printArray(v);
  return 0;
}
```

mergeSort.cpp - merge() function

```
// merge piecewise sorted a[p..q] a[q+1..r] into a sorted a[p..r]
void merge(std::vector<int>& a, int p, int q, int r) {
  std::vector<int> aL, aR; // copy a[p..q] to aL and a[q+1..r] to aR
  for(int i=p; i <= q; ++i) aL.push_back(a[i]);
  for(int i=q+1; i <= r; ++i) aR.push_back(a[i]);
  aL.push_back(INT_MAX);  // append additional value to avoid out-of-bound
  aR.push_back(INT_MAX);
  // pick smaller one first from aL and aR and copy to a[p..r]
  for(int k=p, i=0, j=0; k <= r; ++k) {
    if ( aL[i] <= aR[j] ) {
      a[k] = aL[i];
      ++i;
    }
    else {
      a[k] = aR[j];
      ++j;
    }
  }
}
```

mergeSort.cpp - mergeSort() function

```
void mergeSort(std::vector<int>& a, int p, int r) {
  if ( p < r ) {
    int q = (p+r)/2;       // find a point to divide the problem
    mergeSort(a, p, q);    // divide-and-conquer
    mergeSort(a, q+1, r);  // divide-and-conquer
    merge(a, p, q, r);     // combine the solutions
  }
}
```

Time Complexity of Merge Sort

If $n = 2^m$

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = \sum_{i=1}^{m} cn = cmn = cn\log_2(n) = \Theta(n \log_2 n)$$

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For arbitrary n

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + cn & \text{if } n > 1 \end{cases}$$

$$cn\lfloor \log_2 n \rfloor \leq T(n) \leq cn\lceil \log_2 n \rceil$$

$$T(n) = \Theta(n\log_2 n)$$

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user@host:~/> time cat src/sample.input.txt | src/mergeSort > /dev/null
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```

Quicksort

Quicksort Overview

- Worst-case time complexity is $\Theta(n^2)$
- Expected running time is $\Theta(n \log_2 n)$.
- But in practice mostly performs the best

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Divide and conquer algorithm

Divide Parition (rearrange) the array $A[p..r]$ into two subarrays
- Each element of $A[p..q-1] \leq A[q]$
- Each element of $A[q+1..r] \geq A[q]$

Compute the index q as part of this partitioning procedure

Conquer Sort the two subarrays by recursively calling quicksort

Combine Because the subarrays are already sorted, no work is needed to combine them. The entire array $A[p..r]$ is now sorted

Quicksort Algorithm

Algorithm $\textsc{Quicksort}$

Data: array A and indices p and r
Result: $A[p..r]$ is sorted
if $p < r$ **then**
  q $= \textsc{Partition}(A,p,r)$;
  $\textsc{Quicksort}(A,p,q-1)$;
  $\textsc{Quicksort}(A,q+1,r)$;
end

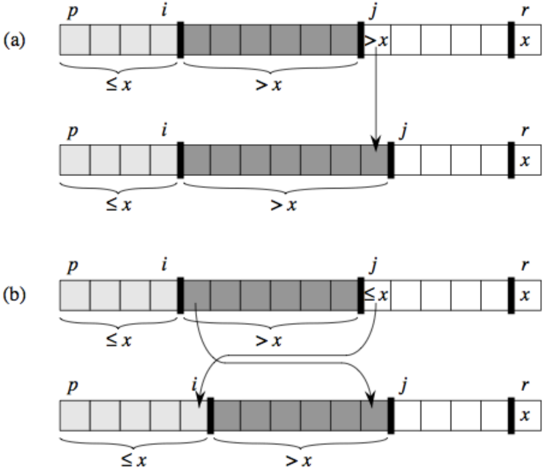
Quicksort Algorithm

Algorithm PARTITION

Data: array A and indices p and r
Result: Returns q such that $A[p..q-1] \leq A[q] \leq A[q+1..r]$

$x = A[r]$;
$i = p - 1$;
for $j = p$ **to** $r - 1$ **do**
    if $A[j] \leq x$ **then**
        $i = i + 1$;
        EXCHANGE($A[i]$,$A[j]$);
    end
end
EXCHANGE($A[i+1]$,$A[r]$);
return $i + 1$;

How PARTITION Algorithm Works



Implementation of QUICKSORT Algorithm

```
// quickSort function
// The main function is the same to mergeSort.cpp except for the function name
void quickSort(std::vector<int>& A, int p, int r) {
  if ( p < r ) { // immediately terminate if subarray size is 1
    int piv = A[r];  // take a pivot value
    int i = p-1;     // p-i-1 is the # elements < piv among A[p..j]
    int tmp;
    for(int j=p; j < r; ++j) {
      if ( A[j] < piv ) { // if smaller value is found, increase q (=i+1)
        ++i;
        tmp = A[i]; A[i] = A[j]; A[j] = tmp; // swap A[i] and A[j]
      }
    }
    A[r] = A[i+1]; A[i+1] = piv;            // swap A[i+1] and A[r]
    quickSort(A, p, i);
    quickSort(A, i+2, r);
  }
}
```

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Lower bounds for comparison sorting

CLRS Theorem 8.1

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An informal proof

- Any comparison sort algorithm can be represented as a binary decision tree, where each node represents a comparison. Each path from the root to leaf represents possible series of comparisons to sort a sequence.

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- We have $n! \leq l \leq 2^h$, where l is the number of leaf nodes, and h is the height of the tree, equivalent to the $\#$ of comparisons.

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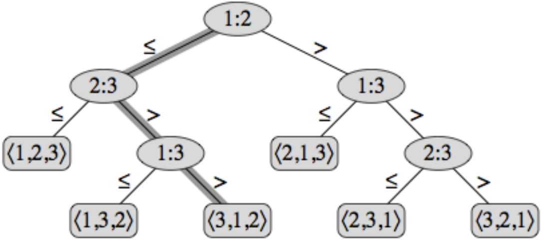
- Each leaf of the decision tree represents one of $n!$ possible permutations of input sequences
- We have $n! \leq l \leq 2^h$, where l is the number of leaf nodes, and h is the height of the tree, equivalent to the $\#$ of comparisons.
- Then it implies $h \geq \log(n!) = \Theta(n \log n)$

Example decision-tree representing InsertionSort



Finding faster sorting methods

Sorting faster than $\Theta(n \log n)$

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Linear time sorting algorithms

- Counting sort
- Radix sort
- Bucket sort

A linear sorting algorithm : Counting sort

A restrictive input setting

- The input sequences have a finite range with many expected duplication.
- For example, each elements of input sequences is one digit number, and your input sequences are millions.

Key idea

1. Scan through each input sequence and count number of occurrences of each possible input value.
2. From the smallest to largest possible input value, output each value repeatedly by its stored count.

Another linear sorting algorithm : Radix sort

Key idea

- Sort the input sequence from the last digit to the first repeatedly using a linear sorting algorithm such as CountingSort
- Applicable to integers within a finite range

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

Sorting Algorithms

- Insertion Sort : $\Theta(n^2)$, loop invariant property
- Merge Sort : $\Theta(n \log n)$, straightforward divide and conquer, a little memory overhead
- Quicksort : $\Theta(n^2)$ worst case, $\Theta(n \log n)$ expected, partition algorithm, practically fast
- Count Sort : Linear algorithm, may require much memory
- Radix Sort : $\Theta(nk)$ with k digits

Next Lecture

Sorting Algorithms

- Radix Sort

Overview of elementary data structures

- Array
- Sorted array
- Linked list
- Binary search tree
- Hash table