Biostatistics 602 - Statistical Inference
Lecture 26
Final Exam Review & Practice Problems for the Final

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April 23rd, 2013

Review of the second half

Rao-Blackwell :

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Rao-Blackwell : If $W(\mathbf{X})$ is an unbiased estimator of $\tau(\theta)$, $\phi(T) = E[W(\mathbf{X})|T]$ is a better unbiased estimator for a sufficient statistic.

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How to get UMVUE Strategies to obtain best unbiased estimators:

- Condition a simple unbiased estimator on complete sufficient statistics
- Come up with a function of sufficient statistic whose expected value is $\tau(\theta)$.

Bayesian Framework

Prior distribution $\pi(\theta)$

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Bayes Estimator is a posterior mean of θ : $E[\theta|\mathbf{x}]$.

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Bayes Rule Estimator minimizes Bayes risk $\Longleftrightarrow$ minimizes posterior expected loss.

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Asymptotic Efficiency of MLE Theorem 10.1.12 MLE is always asymptotically efficient under regularity condition.

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LRT based on sufficient statistics LRT based on full data and sufficient statistics are identical.

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Karlin-Rabin If T is sufficient and has MLR, then test rejecting $R = \{T : T > t_0\}$ or $R = \{T : T < t_0\}$ is an UMP level α test for one-sided composite hypothesis.

Asymptotic Tests and p-Values

Asymptotic Distribution of LRT For testing, $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$,

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Constructing p-Value Theorem 8.3.27 : If large $W(\mathbf{X})$ value gives evidence that H_1 is true, $p(\mathbf{x}) = \sup_{\theta \in \Omega_0} \Pr(W(\mathbf{X}) \geq W(\mathbf{x})|\theta)$ is a valid p-value

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p-Value given sufficient statistics For a sufficient statistic $S(\mathbf{X})$, $p(\mathbf{x}) = \Pr(W(\mathbf{X}) \geq W(\mathbf{x})|S(\mathbf{X}) = S(\mathbf{x}))$ is also a valid p-value.

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Inverting a level α test If $A(\theta_0)$ is the acceptance region of a level α test, then $C(\mathbf{X}) = \{\theta : \mathbf{X} \in A(\theta)\}$ is a $1-\alpha$ confidence set (or interval).

Practice Problem 1 (continued from last week)

Problem

Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2} \qquad -\infty < x < \infty,\ -\infty < \theta < \infty$$

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(c) Show that the test in part (b) is UMP size α for testing $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$.

Solution for (a)

For $\theta_1 < \theta_2$,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\frac{e^{(x-\theta_2)}}{(1+e^{(x-\theta_2)})^2}}{\frac{e^{(x-\theta_1)}}{(1+e^{(x-\theta_1)})^2}}$$

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$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \frac{\frac{e^{(x-\theta_2)}}{(1+e^{(x-\theta_2)})^2}}{\frac{e^{(x-\theta_1)}}{(1+e^{(x-\theta_1)})^2}} = e^{(\theta_1-\theta_2)}\left(\frac{1+e^{(x-\theta_1)}}{1+e^{(x-\theta_2)}}\right)^2$$

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$$r'(x) = \frac{e^{(x-\theta_1)}(1+e^{(x-\theta_2)}) - (1+e^{(x-\theta_1)})e^{(x-\theta_2)}}{(1+e^{(x-\theta_2)})^2} = \frac{e^{(x-\theta_1)} - e^{(x-\theta_2)}}{(1+e^{(x-\theta_2)})^2} > 0 \qquad (\because x-\theta_1 > x-\theta_2)$$

Therefore, the family of X has an MLR.

Solution for (b)

The UMP test rejects H_0 if and only if

$$\frac{f(x|1)}{f(x|0)} = e\left(\frac{1+e^x}{1+e^{(x-1)}}\right)^2 > k$$

Solution for (b)

The UMP test rejects H_0 if and only if

$$\begin{aligned}\frac{f(x|1)}{f(x|0)} &= e\left(\frac{1+e^x}{1+e^{(x-1)}}\right)^2 > k \\ \frac{1+e^x}{1+e^{(x-1)}} &> k^*\end{aligned}$$

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Because under H_0, $F(x_0|\theta = 0) = \frac{e^x}{1+e^x}$, the rejection region of UMP level α test satisfies

$$\begin{aligned}
1 - F(x|\theta = 0) &= \frac{1}{1+e^{x_0}} = \alpha \\
x_0 &\sim \log\left(\frac{1-\alpha}{\alpha}\right)
\end{aligned}$$

Solution for (c)

Because the family of X has an MLR, UMP size α for testing $H_0 : \theta \leq 0$ vs. $H_1 : \theta > 0$ should be a form of

$$\begin{aligned} X &> x_0 \\ \Pr(X > x_0 \mid \theta = 0) &= \alpha \end{aligned}$$

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Therefore, $x_0 = \log\left(\frac{1-\alpha}{\alpha}\right)$, which is identical to the test defined in (b).

Practice Problem 2

Problem

Suppose $X_1, \cdots, X_n$ are iid random samples with pdf $f_X(x|\theta) = \theta \exp(-\theta x)$, where $x \geq 0, \theta > 0$

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(d) Find an asymptotic $(1 - \alpha)$ confidence interval for θ by inverting the above test

You may use the fact that $\mathrm{E}X = 1/\theta$ and $\mathrm{Var}(X) = 1/\theta^2$.

Solution (a) - Consistency

1. Obtain $\mathrm{E}X = 1/\theta$ (Derive yourself if not given)

$$\mathrm{E}X = \int_0^\infty x f(x|\theta)\, dx = \int_0^\infty \theta x \exp(-\theta x)\, dx$$

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2. By LLN (Law of Large Number), $\overline{X} \xrightarrow{\mathrm{P}} \mathrm{E}X = 1/\theta$.
3. By Theorem of continuous map, $n/\sum_{i=1}^n X_i = 1/\overline{X} \xrightarrow{\mathrm{P}} \theta$.

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$$\frac{\sum X_i}{n} = 1/\overline{X} = g(\overline{X}) \sim \mathcal{AN}\left(g(1/\theta), \frac{[g'(1/\theta)]^2}{\theta^2 n}\right)$$

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Solution (c) - Wald asymptotic size α test

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$$S^2 = \frac{n}{\sum_{i=1}^n (X_i - \overline{X})^2} \xrightarrow{\mathrm{P}} \theta^2 \qquad \text{(Slutsky's Theorem).}$$

Solution (c) - Wald Asymptotic size α test (cont'd)

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&= \left| \frac{\frac{n}{\sum_{i=1}^n X_i} - \theta_0}{\frac{1}{\sqrt{\sum_{i=1}^n (X_i - \overline{X})^2}}} \right|
\end{aligned}
$$

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Solution (d) - Asymptotic $1-\alpha$ confidence interval

The acceptance region is

$$A = \left\{ \mathbf{x} : \left| \frac{1}{\overline{x}} - \theta_0 \right| \sqrt{\sum_{i=1}^{n} (x_i - \overline{x})^2} \leq z_{\alpha/2} \right\}$$

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which is equivalent to

$$C(\mathbf{X}) = \left\{ \theta \in \left[\frac{1}{\overline{\overline{X}}} - \frac{z_{\alpha/2}}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X})^2}}, \frac{1}{\overline{\overline{X}}} + \frac{z_{\alpha/2}}{\sqrt{\sum_{i=1}^{n} (X_i - \overline{X})^2}} \right] \right\}$$

Practice Problem 3

Problem

The independent random variables $X_1, \cdots, X_n$ have the following pdf

$$f(x|\theta, \beta) = \frac{\beta x^{\beta-1}}{\theta^\beta} \qquad 0 < x < \theta, \ \beta > 0$$

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2. When β is a known constant β_0, construct a LRT testing $H_0 : \theta \geq \theta_0$ vs. $H_1 : \theta < \theta_0$.

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1. Find the MLEs of β and θ
2. When β is a known constant β_0, construct a LRT testing $H_0 : \theta \geq \theta_0$ vs. $H_1 : \theta < \theta_0$.
3. When β is a known constant β_0, find the upper confidence limit for θ with confidence coefficient $1 - \alpha$.

(a) - MLE

$$L(\theta, \beta|\mathbf{x}) = \frac{\beta^n \left(\prod_{i=1}^n x_i\right)^{\beta-1}}{\theta^{n\beta}} I(x_{(n)} \leq \theta)$$

(a) - MLE

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\hat{\beta} &= \frac{n}{n \log \hat{\theta} - \sum \log x_i} \\
&= \frac{n}{n x_{(n)} - \sum \log x_i}
\end{aligned}$$

(b) - LRT

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Omega_0} L(\hat{\theta}|\mathbf{x})}{\sup_{\theta \in \Omega} L(\hat{\theta}|\mathbf{x})}$$

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$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Omega_0} L(\hat{\theta}|\mathbf{x})}{\sup_{\theta \in \Omega} L(\hat{\theta}|\mathbf{x})} = \begin{cases} 1 & \theta_0 < x_{(n)} \\ \frac{L(\theta_0|\mathbf{x})}{L(x_{(n)}|\mathbf{x})} & \theta_0 \geq x_{(n)} \end{cases}$$

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\lambda(\mathbf{x}) &= \frac{\sup_{\theta \in \Omega_0} L(\hat{\theta}|\mathbf{x})}{\sup_{\theta \in \Omega} L(\hat{\theta}|\mathbf{x})} \\
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&= \begin{cases} 1 & \theta_0 < x_{(n)} \\ \frac{\left(x_{(n)}\right)^{n\beta_0}}{\theta_0^{n\beta_0}} & \theta_0 \geq x_{(n)} \end{cases} \leq c
\end{aligned}$$

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$$\frac{x_{(n)}}{\theta_0} \leq c^*$$

(b) - size α LRT

$$\alpha = \Pr\left(\frac{x_{(n)}}{\theta_0} \leq c^*\right)$$

(b) - size α LRT

$$\begin{aligned}\alpha &= \Pr\left(\frac{x_{(n)}}{\theta_0} \leq c^*\right) \\ &= (c^*)^{n\beta_0}\end{aligned}$$

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\alpha &= \Pr\left(\frac{x_{(n)}}{\theta_0} \leq c^*\right) \\
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c^* &= \alpha^{\frac{1}{n\beta_0}}
\end{aligned}$$

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\alpha &= \Pr\left(\frac{x_{(n)}}{\theta_0} \leq c^*\right) \\
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Therefore, the rejection region for size α LRT is is

(b) - size α LRT

$$\begin{aligned}\alpha &= \Pr\left(\frac{x_{(n)}}{\theta_0} \leq c^*\right) \\ &= (c^*)^{n\beta_0} \\ c^* &= \alpha^{\frac{1}{n\beta_0}}\end{aligned}$$

Therefore, the rejection region for size α LRT is is

$$R = \left\{\mathbf{x} : x_{(n)} \leq \theta_0 \alpha^{\frac{1}{n\beta_0}}\right\}$$

(c) - Upper $1 - \alpha$ confidence limit

The acceptance region of size α LRT is

(c) - Upper $1-\alpha$ confidence limit

The acceptance region of size α LRT is

$$A(\theta_0) = \left\{ \mathbf{x} : x_{(n)} > \theta_0 \alpha^{\frac{1}{n\beta_0}} \right\}$$

(c) - Upper $1 - \alpha$ confidence limit

The acceptance region of size α LRT is

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By inserting the acceptance region, the $1 - \alpha$ confidence interval becomes

(c) - Upper $1-\alpha$ confidence limit

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Therefore, the upper $1 - \alpha$ confidence limit is $X_{(n)} \alpha^{-\frac{1}{n\beta_0}}$.

Practice Problem 4

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A random sample $X_1, \cdots, X_n$ is drawn from a population $\mathcal{N}(\theta, \theta)$ where $\theta > 0$.

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You may use the following fact: $\text{Var}(X^2) = 4\theta^3 + 2\theta^2$.

(a) - MLE of θ

$$L(\theta|\mathbf{x}) = (2\pi\theta)^{n/2} \exp\left[-\frac{\sum_{i=1}^n (x_i - \theta)^2}{2\theta}\right]$$

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\frac{1}{n}\sum x_i^2 &= \hat{\theta}^2 + \hat{\theta}
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(b) - Asymptotic distribution of MLE

By CLT, Let $W = \frac{1}{n}\sum X_i^2$, then

$$W \sim \mathcal{AN}\left(\mathrm{E}X^2, \frac{\mathrm{Var}(X^2)}{n}\right)$$

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for some function $\sigma^2(\theta)$ and we would like to find $\sigma^2(\theta)$ using the asymptotic distribution of W.

(b) - Asymptotic distribution of MLE (cont'd)

Let $g(y) = y^2 + y$, then $g'(y) = (2y + 1)$ and $g(\hat{\theta}) = W$. Then by the Delta Method, the asymptotic distribution of W can be written as

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\sigma^2(\theta) &= \frac{4\theta^3 + 2\theta^2}{(2\theta+1)^2} = \frac{2\theta^2(2\theta+1)}{(2\theta+1)^2} = \frac{2\theta^2}{2\theta+1}
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Note that you cannot use CR-bound for the asymptotic variance of MLE because the regularity condition does not hold (open set criteria).

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Therefore, $\hat{\theta}$ is more efficient estimator than $\overline{X}$.

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6. Don't forget the materials we have learned, because they are the key topics for your candidacy exam.